

Trapped Radial Oscillations of Gaseous Disks around a Black Hole

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Abstract

In gaseous disks around a black hole, the epicyclic frequency does not increase monotonically inward in the radial direction. The effects of general relativity make it the maximum at $4a$, decrease it inward and eventually make it zero at $3a$, where a is the Schwarzschild radius. This implies that a low-frequency wave excited in the inner disk is trapped and cannot propagate outward much beyond the region of radius $4a$. Characteristics of this trapped oscillation are examined under some approximations. For typical values of parameters, the oscillation period of 100 d, which is a typical observed period of time variations of QSOs and Seyfert galaxies, is realized when the mass of the black hole is 10^9 – $10^{10} M_\odot$.

Key words: Accretion disks; Black holes; QSOs; Schwarzschild radius; Trapped oscillations.

1. Introduction

Among various models of QSOs, one of the most promising and extensively examined ones is the gaseous accretion disk around a supermassive black hole. A crucial point to judge the validity of this model is whether the model can explain naturally the periodic (~ 100 d) and quasi-periodic (\sim a few years) time variations observed in some QSOs and Seyfert galaxies. Ozernoy and Usov (1977) emphasized that the difficulty of explaining these time variations by the accretion disk models is one of the serious drawbacks of the models, and gave a preference to the massive magnetoid models.

The purpose of this paper is to show that the periodic time variations can be expected in the gaseous disks. There exist trapped nearly-radial oscillation modes in the inner disk whose radius is $3a$ – $4a$, where a is the Schwarzschild radius.

As is well known, the local frequency of axially-symmetric radial oscillations of cold disks is the epicyclic frequency. In the non-relativistic Keplerian disks, the frequency is the same as the angular frequency of the disk rotation. This means that the frequency increases monotonically inward. If the distance from the black hole is so short that the relativistic effects become non-negligible, however, the situations are changed. The epicyclic frequency does not increase monotonically inward. After arriving at the maximum at $4a$, the frequency decreases and eventually becomes zero at $3a$. This characteristic is related to

the fact that the circular orbit of a particle is unstable inside $3a$ (e.g., Misner 1969).

The presence of the maximum frequency at $4a$ in the epicyclic frequency implies that the lower-frequency oscillations excited in the region inside $4a$ cannot propagate outward far beyond $4a$. In other words, there are long-period trapped oscillations which oscillate only in the inner disk. Studies of the oscillations are worthwhile for the following reasons. The excitation of the oscillations will be expected by the very fact of trapping, and by the possible presence of pulsational instability (Kato 1978, 1979). Second, the oscillations will be observable, because the inner disk is the place from which most of radiation comes (Shakura and Sunyaev 1973). Third, the frequency is so low that too much mass of a black hole is not required to explain the observed periodic oscillations of QSOs.

In section 2, basic equations and the unperturbed state are discussed. In sections 3 and 4, equations describing small amplitude, nearly-radial oscillations are derived and reduced to forms relevant to the study of trapped oscillations. Eigen-frequencies and eigen-functions of the oscillations are obtained in section 5. Section 6 is devoted to discussion.

2. Basic Equations and the Unperturbed State

The square of the invariant line element ds^2 is written in the form $ds^2 = g_{ik}dx^i dx^k$, where $g_{00} > 0$ and $i, k = 0, 1, 2, 3$. Considering a perfect fluid, we write the energy momentum tensor as

$$T^{ik} = (\varepsilon + p)u^i u^k - pg^{ik}, \quad (2.1)$$

where

$$\varepsilon = \rho c^2 + \frac{1}{\gamma - 1} p \quad (2.2)$$

and γ is the ratio of the specific heats. The other notations have their usual meaning.

From the identity $T^{ik}_{;k} = 0$, we get the relativistic hydrodynamic equations:

$$(\varepsilon + p)u^k u^i_{;k} = g^{ik} \frac{\partial p}{\partial x^k} - u^i u^k \frac{\partial p}{\partial x^k} \quad (2.3)$$

and from $(\rho u^i)_{;i} = 0$, the equation of continuity:

$$\frac{1}{(-g)^{1/2}} \frac{\partial}{\partial x^i} [(-g)^{1/2} \cdot \rho u^i] = 0. \quad (2.4)$$

A combination of equations (2.2)–(2.4) gives the adiabatic relation:

$$\frac{1}{(-g)^{1/2}} \frac{\partial}{\partial x^i} \left[(-g)^{1/2} \frac{p}{\gamma - 1} u^i \right] + \frac{1}{(-g)^{1/2}} p \frac{\partial}{\partial x^i} [(-g)^{1/2} \cdot u^i] = 0. \quad (2.5)$$

Hereafter we shall use the 1-, 2-, and 3-components of equation (2.3), equation (2.4), and equation (2.5) as a set of basic hydrodynamic equations.

The self-gravity of the disk is neglected, and the Schwarzschild metric is adopted around a black hole. After writing the metric in terms of the polar

coordinates (r, θ, φ) , we rewrite it in terms of the cylindrical coordinates (ϖ, φ, z) , where $\varpi = r \sin \theta$ and $z = r \cos \theta$. The Schwarzschild metric written by (t, ϖ, φ, z) is adopted hereafter. The expression is given in appendix 1.

The unperturbed axially-symmetric state is regarded as a disk rotating with an angular velocity Ω . Using the metric given in appendix 1 and equation (2.3), we find that the radial and vertical momentum balances of the disk are given by

$$\frac{\varepsilon_0 + p_0}{c^2} \left[\left(1 + \frac{\Omega^2 \varpi^2}{c^2} \right) \left(1 - \frac{a}{r} \right)^{-1} \frac{GM \varpi}{r^3} - \Omega^2 \varpi \right] + \frac{\partial p_0}{\partial \varpi} = 0 \quad (2.6)$$

and

$$\frac{\varepsilon_0 + p_0}{c^2} \left(1 + \frac{\Omega^2 \varpi^2}{c^2} \right) \left(1 - \frac{a}{r} \right)^{-1} \frac{GM z}{r^3} + \frac{\partial p_0}{\partial z} = 0, \quad (2.7)$$

where a is the Schwarzschild radius $2GM/c^2$, M being the mass of the black hole. The subscript 0 represents the quantities of the unperturbed disk.

Hereafter we shall concentrate our attention to the following cases. The temperature of the disk is not so high that the pressure can be neglected in comparison with the rest-mass energy ($p_0 \ll \rho_0 c^2$), and the disk is geometrically thin ($z^2/\varpi^2 \ll 1$). It is implied in this assumption that the pressure force is negligible in the momentum balances in the radial direction; the gravitational force is almost balanced by the centrifugal force (centrifugal balance). Equations (2.6) and (2.7) are then reduced to

$$\Omega^2(\varpi) = \left(1 - \frac{3a}{2\varpi} \right)^{-1} \frac{GM}{\varpi^3} \quad (2.8)$$

and

$$-\frac{\partial p_0}{\partial z} = \rho_0 \Omega^2 z. \quad (2.9)$$

Equation (2.8) shows that the angular velocity of the disk rotation is faster than that of the Keplerian motions by the factor $(1 - 3a/2\varpi)^{-1/2}$. Equation (2.9) shows that in the vertical direction the pressure force is balanced by the gravitational force.

3. Equations of Perturbations

A small perturbation $(u^t, u^\varpi, u^\varphi, u^z, \rho_1, p_1)$ over the above-mentioned equilibrium state is considered as

$$\left. \begin{aligned} u^0 &= \frac{1}{c} \left(1 - \frac{a}{r} \right)^{-1/2} \left(1 + \frac{\Omega^2 \varpi^2}{c^2} \right)^{1/2} + u^t, & u^1 &= \frac{1}{c} u^\varpi, \\ u^2 &= \frac{1}{c} \Omega(\varpi, z) + \frac{1}{c\varpi} u^\varphi, & u^3 &= \frac{1}{c} u^z, \\ \rho &= \rho_0(\varpi, z) + \rho_1, & p &= p_0(\varpi, z) + p_1, \end{aligned} \right\} \quad (3.1)$$

where the unperturbed part of u^0 has been obtained from the unperturbed part of u^2 by use of the identity $u^i u_i = 1$. Substitution of equations (3.1) into equations

(2.3)–(2.5), and linearization of the resulting equations with respect to the perturbed quantities give hydrodynamic equations for small perturbations. These equations are given in appendix 2.

As mentioned before, we adopt $p_0/\rho_0 c^2 \ll 1$ and $z^2/\bar{\omega}^2 \ll 1$ and furthermore adopt equation (2.8) as $\Omega(\bar{\omega})$. Then, for axially-symmetric perturbations, i.e., $\partial/\partial\varphi=0$, the above-mentioned hydrodynamic equations are reduced to simpler forms, which are also given in appendix 2.

We introduce here the so-called “local approximations.” Namely, the characteristic radial scale λ of oscillations is taken to be shorter than any characteristic radial scale in the unperturbed disk. Since we are interested in trapped oscillations, λ is much shorter than the characteristic radius of the disk, say $3a$. Thus, the quantities of the order of $\lambda/3a$ are safely neglected in comparison with unity. The radial scale ($\sim L$) of density and pressure inhomogeneities in the unperturbed disk, however, may not be negligible compared with λ . Actually, in standard accretion disk models, density and pressure change appreciably near $\bar{\omega}=3a$ (e.g. Shakura and Sunyaev 1973). Thus, the inclusion of the effects of density and pressure inhomogeneities in the study of trapped oscillations is important. In this paper, however, we shall not consider such a problem for simplicity, and take $\lambda/L \ll 1$.

The use of the above approximations and the elimination of p_1 and ρ_1 from the hydrodynamic equations given in appendix 2 [namely, equations (A.11)–(A.14)] lead finally to

$$\begin{aligned} & \rho_0 \left(1 - \frac{3a}{2\bar{\omega}_m}\right)^{-1} \left(1 - \frac{a}{\bar{\omega}_m}\right)^{-1} \left[\frac{\partial^2}{\partial t^2} + \left(1 - \frac{3a}{2\bar{\omega}_m}\right) \left(1 - \frac{3a}{\bar{\omega}}\right) \Omega_m^2 \right] u^\varpi \\ &= r p_0 \frac{\partial}{\partial \bar{\omega}} \left(\frac{\partial u^\varpi}{\partial \bar{\omega}} + \frac{\partial u^z}{\partial z} \right) + \frac{\partial}{\partial \bar{\omega}} \left(u^z \frac{\partial p_0}{\partial z} \right) \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} & \rho_0 \left[\left(1 - \frac{3a}{2\bar{\omega}_m}\right)^{-1} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial z} \left(\frac{1}{\rho_0} \frac{\partial p_0}{\partial z} \right) \right] u^z \\ &= \frac{\partial}{\partial z} \left[r p_0 \left(\frac{\partial u^\varpi}{\partial \bar{\omega}} + \frac{\partial u^z}{\partial z} \right) \right] - \frac{\partial p_0}{\partial z} \frac{\partial u^\varpi}{\partial z} + \frac{\partial}{\partial z} \left(u^\varpi \frac{\partial p_0}{\partial \bar{\omega}} \right), \end{aligned} \quad (3.3)$$

where the subscript m to $\bar{\omega}$ and Ω denote their mean values in the trapped region. It should be noted that $\bar{\omega}$ in $(1-3a/\bar{\omega})$ in equation (3.2) is not replaced by $\bar{\omega}_m$.

4. Equations Describing Trapped Oscillations

If the disk has zero temperature, equation (3.2) shows that the disk can oscillate locally in the radial direction with the frequency $(1-3a/2\bar{\omega}_m)^{1/2} (1-3a/\bar{\omega})^{1/2} \Omega_m$. Application of this result is limited to a narrow region near $\bar{\omega}=3a$ because of the approximations used. The presence of such local oscillations, however, is not limited to the region near $\bar{\omega}=3a$. After taking $p_0=0$, we see from equation (A.11) that the frequency of such oscillations at $\bar{\omega}$ is given, in general, by

$$\left[\left(1 - \frac{3a}{2\bar{\omega}}\right) \left(1 - \frac{3a}{\bar{\omega}}\right) \right]^{1/2} \Omega(\bar{\omega}) \quad \text{or} \quad \left[\left(1 - \frac{3a}{\bar{\omega}}\right) \frac{GM}{\bar{\omega}^3} \right]^{1/2}. \quad (4.1)$$

The restoring force of this oscillation comes from the rotation of the disk. The meaning of this frequency is clear from the following considerations. Let us suppose a particle rotating circularly at ϖ . If this circular orbit of the particle is perturbed in the orbital plane, the particle may oscillate around the equilibrium orbit. The motion is described by equations (A.2) and (A.3) with no pressure terms, r being taken to be ϖ . The frequency of this motion is found to be nothing but expression (4.1). In this sense, the frequency (4.1) should be called the epicyclic frequency κ . The relation between κ and Ω is

$$\kappa^2 = \left(1 - \frac{3a}{2\varpi}\right)^2 2\Omega \left(2\Omega + \varpi \frac{d\Omega}{d\varpi}\right). \quad (4.2)$$

This is different from the classical case by the factor $(1 - 3a/2\varpi)^2$. The value of the epicyclic frequency is shown in figure 1 as a function of ϖ by the solid curve. An important point to be noted here is that the epicyclic frequency does not increase monotonically with decreasing radius ϖ , unlike the case of Keplerian disks. It has a maximum at $\varpi = 4a$ and decreases inward, and finally becomes zero at $\varpi = 3a$. Inside $3a$, the frequency is pure imaginary. This corresponds to the well-known fact that the last stable circular orbit is $3a$ (e.g., Misner 1969).

If the disk has pressure, we can expect coherent oscillations of the disk. The oscillations, however, can be no longer purely radial, because the pressure

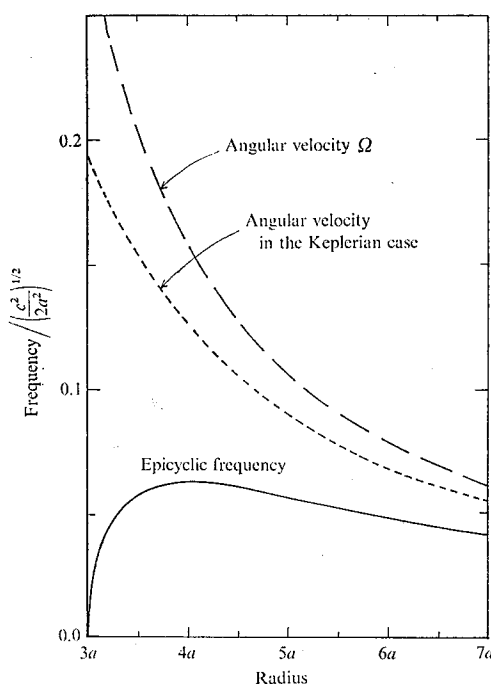


Fig. 1. The radial dependence of the epicyclic frequency κ is shown by the solid curve. The frequency reaches a maximum at $\varpi = 4a$, decreases inward, and eventually becomes zero at $\varpi = 3a$. The broken curve represents the angular velocity of the disk rotation Ω . For comparison, the angular velocity of rotation in the Keplerian case is shown by the dotted curve.

variations associated with the oscillations automatically induces vertical motions. This situation is shown in the set of equations (3.2) and (3.3). That is, $u^{\varpi} \neq 0$ and $u^z = 0$ are no longer solutions. Although the oscillations cannot be purely radial, they are still quasi-radial as long as the pressure force is small compared with the centrifugal force. Furthermore, as long as the pressure force is small in the above sense, the quasi-radial oscillations near $\varpi = 3a$ cannot propagate out far beyond $\varpi = 4a$, because the frequency of such oscillations is lower than the epicyclic frequency at $\varpi = 4a$. That is, the oscillations are trapped inside $4a$ with decreasing amplitude outwards. Our purpose here is to examine such quasi-radial trapped oscillations, under some further approximations.

First, the density distribution in the vertical direction in the unperturbed state is taken to be

$$\rho_0(z) = \rho_{00} \exp(-z^2/2H^2), \quad (4.3)$$

where H is the half thickness of the disk. This implies, since p_0 is related to ρ_0 by equation (2.9), that

$$p_0(z) = p_{00} \exp(-z^2/2H^2), \quad (4.4)$$

where

$$p_{00} = \rho_{00} \Omega^2 H^2. \quad (4.5)$$

Second, the radial velocity u^{ϖ} is approximated to be independent of z , and the vertical velocity u^z to be proportional to z .

The content of the second approximation can be seen by substituting the above z -dependences of u^{ϖ} and u^z into equations (3.2) and (3.3). If it is done, the left-hand side of equation (3.2) is found to be independent of z , but the right-hand side is not so because of the presence of the third term, which is proportional to z^2 . On the other hand, there is no such difficulty in equation (3.3) because both sides are proportional to z . Assuming the above z -dependences of u^{ϖ} and u^z , we integrate equations (3.2) and (3.3) over the z -direction so as to obtain equations for ϖ alone. The results are

$$\begin{aligned} & \left(1 - \frac{3a}{2\varpi_m}\right)^{-1} \left(1 - \frac{a}{\varpi_m}\right)^{-1} \left[\frac{\partial^2}{\partial t^2} + \left(1 - \frac{3a}{2\varpi_m}\right) \left(1 - \frac{3a}{\varpi}\right) \Omega^2 \right] u^{\varpi} \\ & = c_s^2 \frac{\partial}{\partial \varpi} \left(\frac{\partial u^{\varpi}}{\partial \varpi} + \frac{u^z}{z} \right) - \gamma^{-1} c_s^2 \frac{\partial}{\partial \varpi} \left(\frac{u^z}{z} \right) \end{aligned} \quad (4.6)$$

and

$$\left[\left(1 - \frac{3a}{2\varpi_m}\right)^{-1} \frac{\partial^2}{\partial t^2} + (\gamma + 1) \Omega^2 \right] \left(\frac{u^z}{z} \right) = -(\gamma - 1) \Omega^2 \frac{\partial u^{\varpi}}{\partial \varpi}, \quad (4.7)$$

where

$$c_s^2 = \gamma \frac{p_{00}}{\rho_{00}}. \quad (4.8)$$

In deriving the above equations some small-order quantities have been neglected as before.

The above two equations are, of course, not exact because we have used the approximate forms of u^{ϖ} and u^z . The errors, however, are found to be minor and do not bring any qualitative difference in the final results. This can be

seen by tracing forwards the effects of the approximations into the final results.

The elimination of u^2/z from equations (4.6) and (4.7), after taking time variations of perturbed quantities to be $\exp(i\omega t)$, gives an equation of u^w . Taking $\omega^2/\Omega^2 \ll 1$ as well as other approximations used before, we obtain

$$\frac{\partial^2}{\partial \bar{\omega}^2}(\rho_{00}^{1/2} u^w) + P(\bar{\omega}) \rho_{00}^{1/2} u^w = 0, \quad (4.9)$$

where

$$P(\bar{\omega}) = \frac{\gamma(\gamma+1)}{3\gamma-1} c_s^{-2} \left(1 - \frac{a}{\bar{\omega}_m}\right)^{-1} \left(1 - \frac{3a}{2\bar{\omega}_m}\right)^{-1} \left[\omega^2 - \left(1 - \frac{3a}{\bar{\omega}}\right) \frac{GM}{\bar{\omega}_m^3} \right]. \quad (4.10)$$

Since we are interested in the trapped oscillations whose motions are mainly restricted in a narrow region outside $3a$, we write $\bar{\omega}$ in the form

$$\bar{\omega} = 3a + sa \quad (4.11)$$

and small terms s^2 are neglected in comparison with unity. The radius at which $P(\bar{\omega})$ becomes zero is denoted by $\bar{\omega}_c$. The use of equation (4.10) gives

$$\frac{\bar{\omega}_c}{a} - 3 = \frac{6\omega^2}{\Omega^2}. \quad (4.12)$$

Let us write the distance from $\bar{\omega}_c$ by x as

$$x = \bar{\omega}_c - \bar{\omega}. \quad (4.13)$$

Then $P(\bar{\omega})$ is written in the form

$$P(x) = -\frac{\gamma(\gamma+1)}{54(3\gamma-1)} \frac{c_s^2}{c_s^2} \frac{x}{a^3} \equiv \alpha^2 x. \quad (4.14)$$

5. Frequencies of Trapped Oscillations

Our problem here is to solve equation (4.9) with $P(\bar{\omega})$ given by equation (4.14). In the domain of $x > 0$, the two independent solutions of equation (4.9) with respect to $\rho_{00}^{1/2} u^w$ are $x^{1/2} J_{\pm 1/8}(2\alpha x^{3/2}/3)$, while in the domain of $x (= -y) < 0$, they are $y^{1/2} I_{\pm 1/8}(2\alpha y^{3/2}/3)$, where J and I are the Bessel and the modified Bessel functions, respectively.

Since we are interested in the trapped oscillations, u^w should tend to zero for $y \rightarrow \infty$. The characteristics of the modified Bessel functions for large arguments show that this requirement is satisfied when the amplitudes of the two independent solutions $y^{1/2} I_{\pm 1/8}(2\alpha y^{3/2}/3)$ are just opposite in sign. That is, for the region of $x < 0$, we take

$$\rho_{00}^{1/2} u^w = Ay^{1/2} [I_{1/8}(2\alpha y^{3/2}/3) - I_{-1/8}(2\alpha y^{3/2}/3)], \quad (5.1)$$

where $y = -x$, and A is an arbitrary constant. Now we shall extend analytically this solution (5.1) to the region of $x > 0$. Examination of behaviors of the Bessel and the modified Bessel functions shows that $y^{1/2} I_{\pm 1/8}(2\alpha y^{3/2}/3)$ are continued to $\mp x^{1/2} J_{\pm 1/8}(2\alpha x^{3/2}/3)$, respectively. Thus, the solution (5.1) is extended to the region $x \geq 0$ as

$$\rho_{00}^{1/2} u^w = -Ax^{1/2} [J_{1/3}(2\alpha x^{3/2}/3) + J_{-1/3}(2\alpha x^{3/2}/3)] . \quad (5.2)$$

The inner boundary condition we adopt is $\partial u^w / \partial \varpi = 0$ at $\varpi = 3a$. It is not clear what inner boundary condition is most adequate. What we have adopted here is that the Lagrangian variation of pressure is zero at $\varpi = 3a$. This implies $\text{div } \mathbf{u} = 0$ and further is reduced to $\partial u^w / \partial \varpi = 0$ in the quasi-radial oscillations. The above boundary condition is valid when the boundary is a free surface with no pressure. In actual situations, however, the surface of $\varpi = 3a$ will not be such a boundary because of the steady supply of the gas from the region of $\varpi \gtrsim 3a$. To obtain a more reasonable boundary condition, we must examine the physical situation in the region of $\varpi \lesssim 3a$.

Application of the boundary condition $\partial u^w / \partial \varpi$ at $\varpi = 3a$ to equation (5.2) gives

$$J_{2/3}(2\alpha x_{3a}^{3/2}/3) = J_{-2/3}(2\alpha x_{3a}^{3/2}/3) , \quad (5.3)$$

where $x_{3a} = \varpi_c - 3a$. The solution of equation (5.3) for the fundamental mode is

$$\frac{x_{3a}}{a} = 3.80 \left(\frac{c_s}{c} \right)^{2/3} , \quad (5.4)$$

where γ has been taken to be $4/3$. Equation (5.4) shows that the width of the

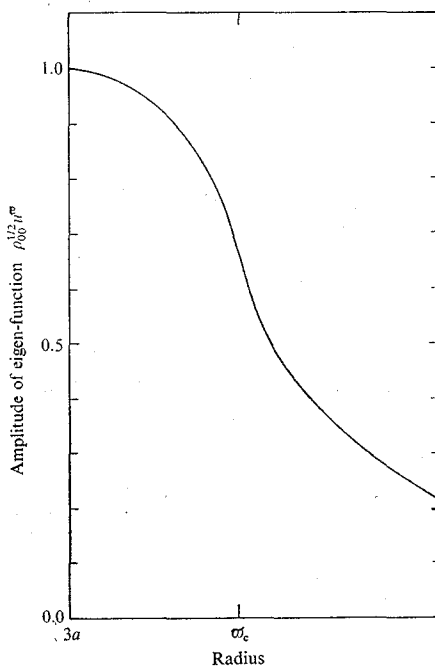


Fig. 2. The eigen-function $\rho_{00}^{1/2} u^w$ of the trapped oscillations. The amplitude is normalized so that it becomes unity at $\varpi = 3a$. The radius ϖ_c , which characterizes the width of the trapped region, depends on c_s , but the figure is drawn by changing the radial scale so that ϖ_c is at a fixed position. Then, the form of eigen-function is independent of c_s .

trapped region depends on the magnitude of the acoustic velocity in the medium. When it is larger, the width becomes wider.

The form of the eigen-function specified by equations (5.1) and (5.2) is shown in figure 2. The figure is drawn so that the point ϖ_0 is at a fixed point on the horizontal axis. Then, the form of eigen-functions is independent of c_s .

The substitution of equation (5.4) into equation (4.12) gives the eigen-frequency ω of the oscillations as

$$\frac{\omega^2}{c^2/2a^2} = 0.047 \left(\frac{c_s}{c} \right)^{2/3}. \quad (5.5)$$

6. Discussion

It is well known that the circular orbit around a black hole is dynamically unstable when $\varpi < 3a$. This is because the general relativistic effects decrease the restoring force to perturbations when ϖ approaches $3a$ from outside and eventually make it vanish at $\varpi = 3a$. This means that in the region just outside of $\varpi = 3a$, the frequency of oscillations around the circular orbit is low. Because of this relativistic effect, we have the trapped oscillations. Their frequency is given by equation (5.5), and the characteristic width x_{3a} of the oscillating region is given by equation (5.4).

The results (5.4) and (5.5), however, can be obtained in the correct order of magnitude, without making detailed calculations. For simplicity, x_{3a} is taken to be a quarter of the wavelength. The frequency ω of acoustic oscillations with such a wavelength is $\omega \sim (\pi/2)c_s/x_{3a}$. The waves can have the trapped nature when the above frequency is equal to the epicyclic frequency at $\varpi = 3a + x_{3a}$, which is $(x_{3a}/3a)^{1/2}(c^2/54a^2)^{1/2}$. Equating these two frequencies, we have

$$\frac{x_{3a}}{3a} \sim \left(\frac{3}{2} \pi^2 \right)^{1/3} \left(\frac{c_s}{c} \right)^{2/3}. \quad (6.1)$$

This is qualitatively the same as equation (5.4). Substitution of x_{3a} given above into $\omega \sim (\pi/2)c_s/x_{3a}$ leads to

$$\frac{\omega^2}{c^2/2a^2} \sim \frac{1}{9} \left(\frac{\pi^2}{18} \right)^{1/3} \left(\frac{c_s}{c} \right)^{2/3}. \quad (6.2)$$

This is also qualitatively the same as equation (5.5).

Equation (5.5) shows that the period $P (= 2\pi/\omega)$ of the trapped oscillation is

$$\frac{P}{10^7 \text{ s}} = 4.0 \times 10^{-8} \frac{M}{10^8 M_\odot} \left(\frac{c_s}{c} \right)^{-1/3}. \quad (6.3)$$

As a typical value we take $c_s/c = 10^{-3}$ and $P = 10^7 \text{ s}$. Equation (6.3) then shows that the mass of the black hole is $M = 2.5 \times 10^8 M_\odot$. The mass range allowed by equation (6.3) is rather narrow. For example, the mass $10^9 M_\odot$ will be unacceptable in the case of $P = 10^7 \text{ s}$, because c_s/c corresponding to it is too low as $c_s/c \sim 6 \times 10^{-5}$.

We shall now examine what is implied on the physical state of the disk by the adoption of $M \sim 2.5 \times 10^8 M_\odot$ derived from equation (6.3). The radial velocity v_ϖ of gaseous inflow due to viscous angular momentum transport is of the order

of $\eta/\rho_0\omega$, where η is the viscosity. Considering turbulent viscosity, we take $\eta \sim \rho_0 c_s H$, where H is the half thickness of the disk. The accretion rate \dot{M} is then given by $\dot{M} \sim 4\pi\omega\rho_0 v_\omega H \sim 4\pi\rho_0 c_s^3/\Omega^2$, where equation (4.5) has been used to rewrite H . This implies that

$$\rho_0 \sim 7.1 \times 10^{-15} \left(\frac{c_s}{c}\right)^{-3} \left(\frac{\dot{M}}{M_\odot/\text{yr}}\right) \left(\frac{M}{10^8 M_\odot}\right)^{-2} \text{ g cm}^{-3}. \quad (6.4)$$

The half thickness H of the disk is expressed by use of equations (4.5) and (4.8) as

$$H \sim 1.3 \times 10^{14} \left(\frac{c_s}{c}\right) \left(\frac{\dot{M}}{10^8 M_\odot}\right) \text{ cm}. \quad (6.5)$$

The accretion rate \dot{M} is taken to be $50 M_\odot \text{ yr}^{-1}$ so that it can explain the luminosity of QSOs. Then, equations (6.4) and (6.5) give $\rho_0 \sim 0.6 \times 10^{-9} \text{ g cm}^{-3}$ and $H \sim 3.4 \times 10^{12} \text{ cm}$. Calculation of temperature from the above c_s/c and ρ_0 gives $T \sim 6.3 \times 10^5 \text{ K}$. The radiation pressure is larger than the gas pressure by some factor. A further estimation by use of H shows that the disk is optically thick and non-self-gravitating.

One may worry about the following points: The trapped region is rather narrow to explain the amplitude of observed light variations, and the required mass of the black hole is somewhat larger than the usually supposed one such as $\sim 10^8 M_\odot$. The above uncomfortable situations, however, will be relaxed, if we consider the following fact. The inner region of luminous disks will be roughly characterized by the constancy of the specific angular momentum, implying that the outward increase of the epicyclic frequency in the region $\omega \gtrsim 3a$ is slower than that shown in figure 1. In a disk with a high accretion rate, the inner part is geometrically thick because of high temperature. The gas is thus pushed into $\omega \lesssim 3a$ from outside by the pressure force rather than by the viscous angular momentum transport. Small pressure force is sufficient to do so because $\partial(\Omega\omega^2)/\partial\omega \approx 0$. Thus, in the inner region of hot disks the specific angular momentum will be nearly constant in space and the epicyclic frequency is low in a wide region, making the trapped region wide. The disk models with constant specific angular momentum have been examined by Lynden-Bell (1978) and Jaroszyński et al. (1980).

It should be noted that the above kind of trapped oscillations will not be restricted only to the galactic nuclei. In the binary systems such oscillations will be expected if the same situations (a black hole and an accretion disk) occur. If we take $M = 1 M_\odot$ and $c_s/c = 10^{-3}$, equation (6.3) shows that the oscillation period is 4 ms.

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Appendix 1. The Metric Used in This Paper

We adopt the Schwarzschild metric. After writing it by the polar coordinates (r, θ, φ) , we introduce the cylindrical coordinates (ϖ, φ, z) , where $\varpi = r \sin \theta$ and $z = r \cos \theta$. Then the square of invariant line element ds^2 is written as

$$ds^2 = \left(1 - \frac{a}{r}\right) c^2 dt^2 - \left(1 - \frac{a}{r}\right)^{-1} \left[\left(1 - \frac{a}{r} \frac{z^2}{r^2}\right) d\varpi^2 + \left(1 - \frac{a}{r}\right) \varpi^2 d\varphi^2 + \left(1 - \frac{a}{r} \frac{\varpi^2}{r^2}\right) dz^2 + 2 \frac{a}{r} \frac{\varpi z}{r^2} d\varpi dz \right]. \quad (\text{A.1})$$

Appendix 2. Equations for Small Perturbations

Substitution of equations (3.1) into equations (2.3)–(2.5) under the use of the metrics given in appendix 1 leads to hydrodynamic equation for perturbations. The linearized forms of these equations are summarized. The equations of motions are

$$\frac{1}{c^2} (\varepsilon + p)_0 \left[\frac{du^\varpi}{dt} - \left(1 - \frac{3a}{2r} \frac{\varpi^2}{r^2}\right) 2\Omega u^\varphi \right] = \frac{(\varepsilon + p)_1}{(\varepsilon + p)_0} \frac{dp_0}{d\varpi} - \frac{dp_1}{d\varpi}, \quad (\text{A.2})$$

$$\frac{1}{c^2} (\varepsilon + p)_0 \left[\frac{du^\varphi}{dt} + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} (\Omega \varpi^2) u^\varpi \right] + \frac{\Omega \varpi}{c^2} \frac{Dp}{Dt} = - \frac{\partial p_1}{\varpi \partial \varphi}, \quad (\text{A.3})$$

and

$$\frac{1}{c^2} (\varepsilon + p)_0 \left[\frac{du^z}{dt} + 3\Omega \frac{a}{r} \frac{\varpi z}{r^2} u^\varphi \right] = \frac{(\varepsilon + p)_1}{(\varepsilon + p)_0} \frac{dp_0}{dz} - \frac{dp_1}{dz}. \quad (\text{A.4})$$

The equation of continuity is

$$\frac{1}{\rho_0} \frac{D\rho}{Dt} + \frac{\Omega \varpi}{c^2} \left(1 - \frac{a}{r}\right)^{-1/2} \left(1 + \frac{\Omega^2 \varpi^2}{c^2}\right)^{-1/2} \frac{\partial u^\varphi}{\partial t} + \left[\frac{1}{\varpi} \frac{\partial}{\partial \varpi} (\varpi u^\varpi) + \frac{\partial u^\varphi}{\varpi \partial \varphi} + \frac{\partial u^z}{\partial z} \right] = 0. \quad (\text{A.5})$$

The adiabatic relation is

$$\frac{Dp}{Dt} + \gamma p_0 \frac{\Omega \varpi}{c^2} \left(1 - \frac{a}{r}\right)^{-1/2} \left(1 + \frac{\Omega^2 \varpi^2}{c^2}\right)^{-1/2} \frac{\partial u^\varphi}{\partial t} + \gamma p_0 \left[\frac{1}{\varpi} \frac{\partial}{\partial \varpi} (\varpi u^\varpi) + \frac{\partial u^\varphi}{\varpi \partial \varphi} + \frac{\partial u^z}{\partial z} \right] = 0, \quad (\text{A.6})$$

where

$$\frac{d}{dt} \equiv \left(1 - \frac{a}{r}\right)^{-1/2} \left(1 + \frac{\Omega^2 \varpi^2}{c^2}\right)^{1/2} \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi}, \quad (\text{A.7})$$

$$\frac{d}{d\varpi} \equiv \left(1 - \frac{a}{r} \frac{\varpi^2}{r^2}\right) \frac{\partial}{\partial \varpi} - \frac{a}{r} \frac{\varpi z}{r^2} \frac{\partial}{\partial z}, \quad (\text{A.8})$$

$$\frac{d}{dz} \equiv \left(1 - \frac{a}{r} \frac{z^2}{r^2}\right) \frac{\partial}{\partial z} - \frac{a}{r} \frac{\varpi z}{r^2} \frac{\partial}{\partial \varpi}, \quad (\text{A.9})$$

$$\frac{D\rho}{Dt} \equiv \frac{d\rho_1}{dt} + \left(u^\varpi \frac{\partial}{\partial \varpi} + u^z \frac{\partial}{\partial z}\right) \rho_0, \quad \frac{Dp}{Dt} \equiv \frac{dp_1}{dt} + \left(u^\varpi \frac{\partial}{\partial \varpi} + u^z \frac{\partial}{\partial z}\right) p_0, \quad (\text{A.10})$$

and the subscripts 0 and 1 represent the unperturbed and perturbed quantities, respectively.

Axially symmetric perturbations are considered under the approximations $p_0 \ll \rho_0 c^2$ and $z^2/\bar{\omega}^2 \ll 1$. The above hydrodynamic equations are then reduced to

$$\begin{aligned} \rho_0 \left(1 - \frac{3a}{2\bar{\omega}}\right)^{-1/2} \left[\frac{\partial^2}{\partial t^2} + \left(1 - \frac{3a}{2\bar{\omega}}\right) \left(1 - \frac{3a}{\bar{\omega}}\right) \Omega^2 \right] u^w \\ = \frac{\partial}{\partial t} \left(\frac{\rho_1}{\rho_0} \right) \frac{dp_0}{d\bar{\omega}} - \frac{d}{d\bar{\omega}} \left[p_0 \frac{\partial}{\partial t} \left(\frac{p_1}{p_0} \right) \right] \end{aligned} \quad (\text{A.11})$$

and

$$\begin{aligned} \rho_0 \left(1 - \frac{3a}{2\bar{\omega}}\right)^{-1/2} \left[\frac{\partial^2}{\partial t^2} - \frac{3}{2} \frac{az}{\bar{\omega}^2} \left(1 - \frac{3a}{\bar{\omega}}\right) \Omega^2 \right] u^z \\ = \frac{\partial}{\partial t} \left(\frac{\rho_1}{\rho_0} \right) \frac{dp_0}{dz} - \frac{d}{dz} \left[p_0 \frac{\partial}{\partial t} \left(\frac{p_1}{p_0} \right) \right], \end{aligned} \quad (\text{A.12})$$

where

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\rho_1}{\rho_0} \right) = \left(1 - \frac{3a}{2\bar{\omega}}\right)^{1/2} \left\{ \frac{1}{2c^2} \left(1 - \frac{a}{\bar{\omega}}\right)^{-1} \left(1 - \frac{3a}{\bar{\omega}}\right) \Omega^2 \bar{\omega} u^w \right. \\ \left. - \left[\frac{1}{\bar{\omega}} \frac{\partial}{\partial \bar{\omega}} (\bar{\omega} u^w) + \frac{\partial u^z}{\partial z} \right] - \left(u^w \frac{\partial}{\partial \bar{\omega}} + u^z \frac{\partial}{\partial z} \right) \ln \rho_0 \right\} \end{aligned} \quad (\text{A.13})$$

and

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{p_1}{p_0} \right) = \left(1 - \frac{3a}{2\bar{\omega}}\right)^{1/2} \left\{ \frac{r}{2c^2} \left(1 - \frac{a}{\bar{\omega}}\right)^{-1} \left(1 - \frac{3a}{\bar{\omega}}\right) \Omega^2 \bar{\omega} u^w \right. \\ \left. - r \left[\frac{1}{\bar{\omega}} \frac{\partial}{\partial \bar{\omega}} (\bar{\omega} u^w) + \frac{\partial u^z}{\partial z} \right] - \left(u^w \frac{\partial}{\partial \bar{\omega}} + u^z \frac{\partial}{\partial z} \right) \ln p_0 \right\}. \end{aligned} \quad (\text{A.14})$$

The operator $d/d\bar{\omega}$ and d/dz are also reduced to

$$\frac{d}{d\bar{\omega}} = \left(1 - \frac{a}{\bar{\omega}}\right) \frac{\partial}{\partial \bar{\omega}} - \frac{az}{\bar{\omega}^2} \frac{\partial}{\partial z} \quad (\text{A.15})$$

and

$$\frac{d}{dz} = \frac{\partial}{\partial z} - \frac{az}{\bar{\omega}^2} \frac{\partial}{\partial \bar{\omega}}. \quad (\text{A.16})$$

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