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Radiation Hydrodynamics in a Moving Plasma with Compton Scattering

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Abstract

We examine the radiative transfer in a hot moving plasma which interacts with photons through the Compton processes. The Klein–Nishina differential cross section is used in the formulation. By assuming that the electron kinetic energy and the photon energy are sufficiently low in comparison with the electron rest mass energy, the effects of the Compton processes are taken into account in the formulation only as small corrections over the Thomson scattering. Moment equations under the diffusion approximation and radiative viscosity in an optically thick plasma are derived in frequency-dependent forms. The coefficient of radiative viscosity slightly increases in the subrelativistic regime in comparison with that under the Thomson scattering. Equations governing the radiation hydrodynamics in the subrelativistic regime are briefly mentioned. The radiation force and the radiation drag on plasma also undergo small changes from those due to the Thomson scattering.

Key words: Compton scattering; High energy astrophysics; Radiation hydrodynamics; Radiative transfer; Radiative viscosity.

1. Introduction

A number of observational facts and theoretical results are now accumulated on high energetic astrophysical phenomena, e.g., X-ray bursters, supernova explosions, astrophysical jets, nuclei of active galaxies, and the early epoch of the universe. Under such circumstances, it becomes more and more necessary to study in detail the interaction processes between radiation field and matter in a hot plasma with a bulk motion.

Since Kompaneets (1957) the Comptonization of the isotropic radiation field in a hot plasma has been extensively investigated [see Sunyaev and Titarchuk (1980) and references therein]. If the change of energy in an individual scattering is small in comparison with the incident photon energy, the evolution of photon distribution is subject to the Kompaneets (1957) equation. Much attention, however, should be paid to its generalization to the cases where the hot plasma has a bulk motion or where the radiation field has anisotropy. The study of such cases is also important for the derivation of hydrodynamical equations describing the hot moving plasma.

The equations of radiative transfer and radiation hydrodynamics are already obtained by Hsieh and Spiegel (1976) in the case where photons and the plasma interact through the Thomson scattering. The next requirement is to extend their work to the case of a hot plasma where photons and the plasma interact through the Compton processes. This problem is partially answered by Blandford and Payne (1981) and Masaki (1981). Their work, however, seems to have some limitations in real applications. As for Blandford and Payne (1981), they introduced the diffusion approximation in their formulation without a careful distinction between the fluid frame and the inertial frame. Furthermore, in their formulation the Klein–Nishina differential cross section is not used. Finally, they did not generalize the work by Hsieh and Spiegel (1976) until radiative viscosity and the equations of radiation hydrodynamics under the Compton scattering are obtained. On the other hand, Masaki (1981) examined the radiative transfer in a hot plasma, taking into account the Compton scattering. His formulation, however, is too general to be used for practical purposes. Also he did not consider the bulk motion of the plasma and the radiation hydrodynamics.

The purpose of this paper is thus to complement their works on the above-mentioned points. Like Blandford and Payne (1981), however, the effects of energy change, Doppler frequency shift, and aberration are taken into account in the formulation only as small corrections to the Thomson scattering.

In the next section, we derive the transfer equation of photons, taking into account the Klein–Nishina differential cross section for the Compton scattering. In section 3, frequency-dependent moment equations of the radiative transfer are presented under the diffusion approximation. In section 4, radiative viscosity in the Thomson scattering (Masaki 1971) is generalized to the case of the Compton scattering. Frequency-integrated moment equations of photons and equations for a hot moving plasma are summarized in section 5. The final section is devoted to discussions.

2. Collision Integrals and Transfer Equations

In this section we derive the photon's transfer equation expressed in the frame comoving with the fluid (*fluid frame*). To do so, however, the collision terms are first described in the frame where the electron interacting with photons is at rest (*electron rest frame*). In the next section, the moments of the transfer equation are taken in the *inertial frame*. In this paper we distinguish definitely these three frames. In what follows, a subscript 0, a bar, and no mark represent respectively quantities measured in the electron rest frame, those in the fluid frame, and those in the inertial frame. Throughout this paper, we shall work in units where $c=h=k=1$; c, h, and k being respectively the light speed, the Planck constant, and the Boltzmann constant.

It is convenient to start from the transfer equation which is written in the form invariant to the Lorentz transformation (Lindquist 1966; Lifshitz and Pitaevskii 1981):

$$k^{\mu}\partial_{\mu}n(\bar{\nu},\bar{\boldsymbol{l}})=\int S(\bar{\nu},\bar{\boldsymbol{l}};\bar{\varepsilon},\bar{\boldsymbol{P}})\cdot f_{\mathrm{e}}\cdot\varepsilon_0\cdot\frac{d^3\bar{\boldsymbol{P}}}{\bar{\varepsilon}}\,, \tag{1}$$

where the bars have been attached to some quantities in order to emphasize that we are interested here in the equation expressed in the fluid frame. Here $k^{\mu}\partial_{\mu}$ is the directional derivative along the photon trajectory in the four-dimensional phase space, k^{μ} being a component of the photon's four-momentum, i.e., $k^{\mu}=\nu(1,\boldsymbol{l})$. ν is the frequency of the photon and $\boldsymbol{l}$ is the unit vector in the photon's direction of motion. In this paper, the Greek suffixes take the values 0, 1, 2, and 3, whereas the Latin suffixes take the values 1, 2, and 3. The quantity $n(\nu,\boldsymbol{l})$ represents the invariant occupation number of photons. The term $S(\bar{\nu},\bar{\boldsymbol{l}};\bar{\varepsilon},\bar{\boldsymbol{P}})$ represents the rate of collisions between a photon with frequency $\bar{\nu}$ and momentum $\bar{\nu}\bar{\boldsymbol{l}}$ and an electron with energy $\bar{\varepsilon}$ and momentum $\bar{\boldsymbol{P}}$. f_{e} is the invariant electron distribution function in the phase space and ε_0 is the electron energy in its rest frame.

The invariant collision integral S may be written in a simpler form in the frame where the electron under consideration ($\bar{\varepsilon}$ and $\bar{\boldsymbol{P}}$) is at rest. In that frame the expression for S is (Hsieh and Spiegel 1976; Blandford and Payne 1981)

$$\begin{aligned} S(\bar{\nu},\bar{\boldsymbol{l}};\bar{\varepsilon},\bar{\boldsymbol{P}})&=S(\nu_0,\boldsymbol{l}_0;\bar{\varepsilon},\bar{\boldsymbol{P}})\\ &=\nu_0[1+n(\nu_0,\boldsymbol{l}_0)]\int d\Omega_0'\frac{d}{d\Omega_0}\sigma(\nu_0',\boldsymbol{l}_0'\to\nu_0,\boldsymbol{l}_0)\left(\frac{\nu_0'}{\nu_0}\right)^2\frac{d\nu_0'}{d\nu_0}n(\nu_0',\boldsymbol{l}_0')\\ &\quad-\nu_0 n(\nu_0,\boldsymbol{l}_0)\int d\Omega_0'[1+n(\nu_0',\boldsymbol{l}_0')]\frac{d}{d\Omega_0'}\sigma(\nu_0,\boldsymbol{l}_0\to\nu_0',\boldsymbol{l}_0')\,. \end{aligned} \tag{2}$$

Here the differential cross section $d\sigma/d\Omega_0$ of the scattering is given by the Klein–Nishina formula:

$$\frac{d}{d\Omega_0}\sigma(\nu_0',\boldsymbol{l}_0'\to\nu_0,\boldsymbol{l}_0)=\frac{3}{16\pi}\sigma_{\mathrm{T}}\left(\frac{\nu_0}{\nu_0'}\right)^2\left[\frac{\nu_0'}{\nu_0}+\frac{\nu_0}{\nu_0'}-1+(\boldsymbol{l}_0\cdot\boldsymbol{l}_0')^2\right], \tag{3}$$

where σ_{T} is the total cross section of the Thomson scattering. The frequency change of the photon from ν_0' to ν_0 due to scattering is given by the Compton formula:

$$\nu_0=\frac{\nu_0'}{1+\dfrac{\nu_0'}{m_{\mathrm{e}}}(1-\boldsymbol{l}_0\cdot\boldsymbol{l}_0')}\,, \tag{4}$$

where $\boldsymbol{l}_0\cdot\boldsymbol{l}_0'$ is the cosine of the scattering angle in the electron rest frame and m_{e} is the electron rest mass.

In equation (2), the term $(\nu_0'/\nu_0)^2 d\nu_0'/d\nu_0$ in the first integrand comes from the fact that higher frequency has larger phase space. The factors $(1+n)$ represent the induced enhancement of scattering due to the presence of photons in the final state. Blandford and Payne (1981) neglected the induced terms from the beginning.

Finally the transformation relations between $(\bar{\nu},\bar{\boldsymbol{l}})$ and $(\nu_0,\boldsymbol{l}_0)$ are necessary in order to close the description of the transfer equation. These Lorentz transformations are (e.g., Hsieh and Spiegel 1976)

$$\nu_0=\Gamma\bar{\nu}(1-\boldsymbol{V}\cdot\bar{\boldsymbol{l}})\ , \tag{5a}$$

$$\boldsymbol{l}_0=\frac{\bar{\nu}}{\nu_0}\left[\bar{\boldsymbol{l}}+\left(\frac{\Gamma-1}{V^2}\boldsymbol{V}\cdot\bar{\boldsymbol{l}}-\Gamma\right)\boldsymbol{V}\right]\ , \tag{5b}$$

where $\Gamma=(1-V^2)^{-1/2}=\bar{P}/m_e V$ and $\boldsymbol{V}$ is the velocity of an individual electron in the fluid frame (thermal velocity). It is easy to see that the transfer equation described by the set of equations (1)–(5) is identical with that given by Sampson (1959) [see also Masaki (1981)].

In this paper, we assume that the plasma is sufficiently subrelativistic both in thermal motion and in bulk motion. That is, in the following formulations, we keep terms up to the linear ones with respect to T/m_e and $\boldsymbol{v}$, where T is the electron temperature and $\boldsymbol{v}$ is the bulk velocity of plasma. The energy of the photon is also assumed to be subrelativistic; the terms up to the first order in ν/m_e are retained. The medium is assumed to be optically thick to the electron scattering. Double Compton scattering and pair processes are neglected as well as emission and absorption.

Under the above approximations, we substitute equations (3) and (4) into equation (2) to yield

$$S=\frac{3}{16\pi}\nu_0\sigma_T[1+n(\nu_0,\boldsymbol{l}_0)]\int d\Omega_0'[1+(\boldsymbol{l}_0\cdot\boldsymbol{l}_0')^2]n(\nu_0+\delta\nu_0,\boldsymbol{l}_0')\frac{\partial}{\partial\nu_0}(\nu_0+\delta\nu_0)$$
$$-\frac{3}{16\pi}\nu_0\sigma_T n(\nu_0,\boldsymbol{l}_0)\int d\Omega_0'\left(\frac{\nu_0-\delta\nu_0}{\nu_0}\right)^2[1+(\boldsymbol{l}_0\cdot\boldsymbol{l}_0')^2][1+n(\nu_0-\delta\nu_0,\boldsymbol{l}_0')]\ , \tag{6}$$

where

$$\delta\nu_0=\frac{\nu_0^2}{m_e}(1-\boldsymbol{l}_0\cdot\boldsymbol{l}_0')\ . \tag{7}$$

The next step is to express the transfer equation in the fluid frame in an explicit form. To do so, we first rewrite the collision term of the transfer equation by means of the quantities in the fluid frame with the help of the Lorentz transformation (5) and then average it over the thermal motion of electrons. The term $n(\nu_0+\delta\nu_0, \boldsymbol{l}_0')$ in equation (6), for example, can be expressed by the quantities in the fluid frame, up to the second order in $\boldsymbol{V}$ (i.e., up to the first order in T/m_e), as

$$n(\nu_0+\delta\nu_0,\boldsymbol{l}_0')=n(\bar{\nu}',\bar{\boldsymbol{l}}')$$
$$\simeq n(\bar{\nu},\bar{\boldsymbol{l}}')+\left[(\bar{\boldsymbol{l}}'-\bar{\boldsymbol{l}})\cdot\boldsymbol{V}+(\boldsymbol{V}\cdot\bar{\boldsymbol{l}}')(\bar{\boldsymbol{l}}'-\bar{\boldsymbol{l}})\cdot\boldsymbol{V}+\frac{\delta\nu_0}{\nu_0}\right]\bar{\nu}\frac{\partial}{\partial\bar{\nu}}n(\bar{\nu},\bar{\boldsymbol{l}}')$$
$$+\frac{1}{2}[(\bar{\boldsymbol{l}}'-\bar{\boldsymbol{l}})\cdot\boldsymbol{V}]^2\bar{\nu}^2\frac{\partial}{\partial\bar{\nu}^2}n(\bar{\nu},\bar{\boldsymbol{l}}')\ . \tag{8}$$

Here the first equality shows the invariance of n. After the above procedures and approximations, we can write the invariant collision integral as

$$S=\frac{3}{16\pi}\nu_0\sigma_T\int d\Omega_0'[1+(\boldsymbol{l}_0\cdot\boldsymbol{l}_0')^2]$$
$$\times\left\{n(\bar{\nu},\bar{\boldsymbol{l}}')+[(\bar{\boldsymbol{l}}'-\bar{\boldsymbol{l}})\cdot\boldsymbol{V}+(\boldsymbol{V}\cdot\bar{\boldsymbol{l}}')(\bar{\boldsymbol{l}}'-\bar{\boldsymbol{l}})\cdot\boldsymbol{V}]\bar{\nu}\frac{\partial}{\partial\bar{\nu}}n(\bar{\nu},\bar{\boldsymbol{l}}')\right.$$

$$+\frac{1}{2}[(\bar{l}'-\bar{l})\cdot V]^2\bar{\nu}^2\frac{\partial}{\partial\bar{\nu}^2}n(\bar{\nu},\bar{l}')$$

$$+\frac{\bar{\nu}}{m_e}(1-\bar{l}\cdot\bar{l}')[1+2n(\bar{\nu},\bar{l})]\left[2n(\bar{\nu},\bar{l}')+\bar{\nu}\frac{\partial}{\partial\bar{\nu}}n(\bar{\nu},\bar{l}')\right]\Bigg\}$$

$$-\left(1-\frac{2\bar{\nu}}{m_e}\right)\nu_0\sigma_T n(\bar{\nu},\bar{l}), \tag{9}$$

where ν_0 and $(l_0\cdot l_0')$ should be expressed in terms of $\bar{\nu}$, $\bar{l}$, $\bar{l}'$, and V with the help of equation (5).

We assume that the electrons have isotropic velocity distribution in the fluid frame. Furthermore, the temperature T of the electron gas is defined by averaging V over the momentum space as $\langle V^2\rangle=3T/m_e$. From equations (1) and (9), after performing the integration over the momentum space, we finally obtain the transfer equation in the fluid frame:

$$k^\mu\partial_\mu n(\bar{\nu},\bar{l})=\left(1-\frac{2\bar{\nu}}{m_e}\right)n_e\bar{\nu}\sigma_T\left[\frac{3}{16\pi}\int d\bar{\Omega}'[1+(\bar{l}\cdot\bar{l}')^2]n(\bar{\nu},\bar{l}')-n(\bar{\nu},\bar{l})\right]$$

$$+\frac{2\bar{\nu}}{m_e}\frac{3}{16\pi}n_e\bar{\nu}\sigma_T\int d\bar{\Omega}'[(\bar{l}\cdot\bar{l}')^3+(\bar{l}\cdot\bar{l}')]n(\bar{\nu},\bar{l}')$$

$$+\frac{2T}{m_e}\frac{3}{16\pi}n_e\bar{\nu}\sigma_T\int d\bar{\Omega}'[2(\bar{l}\cdot\bar{l}')^3-3(\bar{l}\cdot\bar{l}')^2-2(\bar{l}\cdot\bar{l}')+1]n(\bar{\nu},\bar{l}')$$

$$+\frac{3}{16\pi}n_e\bar{\nu}\sigma_T\int d\bar{\Omega}'[1+(\bar{l}\cdot\bar{l}')^2](1-\bar{l}\cdot\bar{l}')$$

$$\times\frac{1}{m_e}\left[\frac{T}{\bar{\nu}^2}\frac{\partial}{\partial\bar{\nu}}\bar{\nu}^4\frac{\partial}{\partial\bar{\nu}}+\frac{1}{\bar{\nu}^2}\frac{\partial}{\partial\bar{\nu}}\bar{\nu}^4+2n(\bar{\nu},\bar{l})\frac{\partial}{\partial\bar{\nu}}\bar{\nu}^2\right]n(\bar{\nu},\bar{l}'), \tag{10}$$

where n_e is the electron number density given by integration of $f_e(\boldsymbol{P})$ over the momentum space. Due to the assumption of the isotropic electron distribution, the odd terms of V vanish automatically after averaging over V. Thus equation (10) is valid up to the order of V^3.

The factor $(1-2\bar{\nu}/m_e)$ in the first term on the right-hand side of equation (10) originates from the Klein–Nishina reduction, where the total cross section of scattering decreases as the energy of the incident photon increases (e.g., Heitler 1954). The term of the second integral comes from the fact that in the Compton scattering the probability distribution of the scattering direction changes from that of the Thomson scattering. The term of the third integral is a small correction expressing the effect of the velocity dispersion of electrons. These terms of the second and the third integrals, as well as the first term, vanish when the photon distribution is isotropic. The term of the last integral describes the evolution of the photon distribution function due to repeated Compton scattering. In fact, in the limit of the isotropic n, equation (10) is reduced just to the Kompaneets (1957) equation.

3. Frequency-Dependent Moment Equations

The transfer equation (10) is too complicated to be applied to many practical problems. Hence, as is done in many cases, we take moments of the equation. To close the moment equations, we adopt here the diffusion approximation in the fluid frame:

$$n(\bar{\nu}, \bar{l}) = \bar{n}(\bar{\nu}) + 3\bar{l} \cdot \bar{f}(\bar{\nu}) , \tag{11}$$

which is rough but still useful as the first approximation in many cases. Here $\bar{n}$ and $\bar{f}$ are the zeroth and first moments of photon's occupation number, and are defined respectively as

$$\bar{n}(\bar{\nu}) \equiv \frac{1}{4\pi} \int d\bar{\Omega} n(\bar{\nu}, \bar{l}) , \tag{12a}$$

$$\bar{f}^i(\bar{\nu}) \equiv \frac{1}{4\pi} \int d\bar{\Omega} \bar{l}^i n(\bar{\nu}, \bar{l}) . \tag{12b}$$

Higher moments are defined similarly as

$$\bar{p}^{ij}(\bar{\nu}) \equiv \frac{1}{4\pi} \int d\bar{\Omega} \bar{l}^i \bar{l}^j n(\bar{\nu}, \bar{l}) , \tag{12c}$$

$$\bar{q}^{ijk}(\bar{\nu}) \equiv \frac{1}{4\pi} \int d\bar{\Omega} \bar{l}^i \bar{l}^j \bar{l}^k n(\bar{\nu}, \bar{l}) , \tag{12d}$$

which will appear later.

Although Blandford and Payne (1981) adopted the diffusion approximation in the inertial frame, it should be done in the fluid frame by this approach (Hsieh and Spiegel 1976).

Substituting equation (11) into equation (10) and performing the integrations in equation (10), we find

$$\begin{aligned} k^\mu \partial_\mu n(\bar{\nu}, \bar{l}) = & -\left(1 - \frac{14\bar{\nu}}{5m_e} + \frac{2T}{5m_e}\right) n_e \bar{\nu} \sigma_T 3\bar{l} \cdot \bar{f} \\ & + \frac{n_e \bar{\nu} \sigma_T}{m_e} \left[\frac{1}{\bar{\nu}^2} \frac{\partial}{\partial \bar{\nu}} \left[\bar{\nu}^4 \left(T \frac{\partial \bar{n}}{\partial \bar{\nu}} + \bar{n} + \bar{n}^2 \right) \right] + 6\bar{l} \cdot \bar{f} \frac{\partial}{\partial \bar{\nu}} (\bar{\nu}^2 \bar{n}) \right. \\ & \left. - \left\{ \frac{1}{\bar{\nu}^2} \frac{\partial}{\partial \bar{\nu}} \left[\bar{\nu}^4 \left(T \frac{\partial}{\partial \bar{\nu}} + 1 \right) \right] + 2(\bar{n} + 3\bar{l} \cdot \bar{f}) \frac{\partial}{\partial \bar{\nu}} \bar{\nu}^2 \right\} \left(\frac{6}{5} \bar{l} \cdot \bar{f} \right) \right] . \end{aligned} \tag{13}$$

This is the transfer equation in the fluid frame under the diffusion approximation. The factor $(1 - 14\bar{\nu}/5m_e + 2T/5m_e)$ in the first term of equation (13) is different from $(1 - 2\bar{\nu}/m_e)$ in equation (10), for the terms of the second and third integrals in equation (10) have some contributions. The terms in large closed brackets come from the last integral in equation (10), which represents the effect of Comptonization. Taking the moments of equation (13), we can easily have the moment equations in the fluid frame. For practical purposes, however, moment equations in the inertial frame are more useful. Hence we shall direct our attention to them in the following.

The Lorentz transformation between the photon's four momentum $\bar{\nu}(1, \bar{l})$ in the fluid frame and $\nu(1, l)$ in the inertial frame are similar to those given by equation (5). Since the bulk velocity $\boldsymbol{v}$ of the plasma is assumed subrelativistic, the Doppler effect and aberration become

$$\bar{\nu}=\nu(1-\boldsymbol{v}\cdot\boldsymbol{l})\,, \tag{14a}$$

$$\bar{\boldsymbol{l}}=\boldsymbol{l}-\boldsymbol{v}+(\boldsymbol{v}\cdot\boldsymbol{l})\boldsymbol{l}\,. \tag{14b}$$

With the help of equation (14) and $n(\bar{\nu}, \bar{\boldsymbol{l}})=n(\nu, \boldsymbol{l})$, we can derive the *transformation rules* among the moments of the photon's occupation number:

$$\bar{n}(\bar{\nu})=n(\nu)-(\boldsymbol{v}\cdot\boldsymbol{l})\nu\frac{\partial}{\partial\nu}n(\nu)+2\boldsymbol{v}\cdot\boldsymbol{f}(\nu)+\nu\frac{\partial}{\partial\nu}[\boldsymbol{v}\cdot\boldsymbol{f}(\nu)]\,, \tag{15}$$

$$\bar{f}^i(\bar{\nu})=f^i(\nu)-(\boldsymbol{v}\cdot\boldsymbol{l})\nu\frac{\partial}{\partial\nu}f^i(\nu)-v^i n(\nu)+3v_j p^{ij}(\nu)+\nu\frac{\partial}{\partial\nu}v_j p^{ij}(\nu)\,, \tag{16}$$

$$\begin{aligned}\bar{p}^{ij}(\bar{\nu})=&\,p^{ij}(\nu)-(\boldsymbol{v}\cdot\boldsymbol{l})\nu\frac{\partial}{\partial\nu}p^{ij}(\nu)-v^i f^j(\nu)-v^j f^i(\nu)\\&+4v_k q^{ijk}(\nu)+\nu\frac{\partial}{\partial\nu}v_k q^{ijk}(\nu)\,.\end{aligned} \tag{17}$$

Here the moments in the inertial frame, $n(\nu)$, $f^i(\nu)$, $p^{ij}(\nu)$, and $q^{ijk}(\nu)$, are defined by equations similar to equation (12). It is noted that equations (15)–(17) are general in the sense that the diffusion approximation is not used.

Substitution of equations (14)–(16) into equation (13) and integration over the solid angle ultimately give the moment equations in the *inertial frame*. The first two moments are

$$\begin{aligned}\frac{\partial n}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{f}=&\left(1-\frac{14\nu}{5m_e}+\frac{2T}{5m_e}\right)n_e\sigma_T\left[3\boldsymbol{v}\cdot\boldsymbol{f}+\nu\frac{\partial}{\partial\nu}(\boldsymbol{v}\cdot\boldsymbol{f})\right]\\&+\frac{n_e\sigma_T}{m_e}\frac{1}{\nu^2}\frac{\partial}{\partial\nu}\left[\nu^4\left(T\frac{\partial n}{\partial\nu}+n+n^2\right)\right]-\frac{n_e\sigma_T}{m_e}\frac{12}{5}\boldsymbol{f}\cdot\frac{\partial}{\partial\nu}(\nu^2\boldsymbol{f})\,,\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{\partial f^i}{\partial t}+\partial_j p^{ij}=&-\left(1-\frac{14\nu}{5m_e}+\frac{2T}{5m_e}\right)n_e\sigma_T\left(f^i+\frac{v^i}{3}\nu\frac{\partial n}{\partial\nu}\right)\\&+\frac{n_e\sigma_T}{m_e}\left\{2f^i\frac{\partial}{\partial\nu}(\nu^2 n)-\frac{4}{5}n\frac{\partial}{\partial\nu}(\nu^2 f^i)-\frac{2}{5\nu^2}\frac{\partial}{\partial\nu}\left[\nu^4\left(T\frac{\partial f^i}{\partial\nu}+f^i\right)\right]\right\}\,,\end{aligned} \tag{19}$$

where n and f^i are functions only of ν. The zeroth moment $n(\nu)$ should not be confused with $n(\nu, \boldsymbol{l})$. We note that the derivatives, $\partial/\partial t$ and $\boldsymbol{\nabla}$, are operated in the inertial frame with fixed ν and $\boldsymbol{l}$.

The moment equations (18) and (19) are to be closed under the diffusion approximation (11). Actually, in the case of the diffusion approximation, the frequency-dependent stress tensor $p^{ij}(\nu)$ on the left-hand side of equation (19) is expressed by means of $n(\nu)$ and $f^i(\nu)$ as

$$\begin{aligned}p^{ij}(\nu)=&\frac{\delta^{ij}}{3}n+\frac{1}{5}\left[v^i f^j+v^j f^i-\nu\frac{\partial}{\partial\nu}(v^i f^j)-\nu\frac{\partial}{\partial\nu}(v^j f^i)\right]\\&-\frac{2}{15}\delta^{ij}\left[v^k f^k-\nu\frac{\partial}{\partial\nu}(v^k f^k)\right].\end{aligned} \tag{20}$$

Here we have used the transformation rules (15) and (16).

Equations (18)–(20) are the closed set of the frequency-dependent moment equations in the inertial frame under the diffusion approximation. If we neglect the terms of induced scattering and other higher-order ones which are small, equations (18)–(20) are reduced to Blandford and Payne's (1981) equations (15) and (16). The term $3\boldsymbol{v}\cdot\boldsymbol{f}$ in equation (18) is dropped in their equation (15) due to misprint. However, terms proportional to $\boldsymbol{v}$ in equation (20) are absent in their results, since they did not distinguish clearly between the fluid frame and the inertial frame. These terms arise from the Lorentz transformation (Hsieh and Spiegel 1976).

When equations (18)–(20) are integrated over frequencies, the equations to be used for radiation hydrodynamics are obtained. Before proceeding in such a direction, we shall discuss radiative viscosity in the next section.

4. Radiative Viscosity

In the previous section we have used the diffusion approximation, $\bar{p}^{ij}(\bar{\nu})=\delta^{ij}\bar{n}(\bar{\nu})/3$ or equation (11), in order to obtain the frequency-dependent moment equations. Under this approximation, however, an important property of interaction between the photon and the fluid, i.e., radiative viscosity, has been dropped from the consideration from the beginning. In this section we shall examine how the radiation stress tensor $\bar{p}^{ij}(\bar{\nu})$ is modified in the next order of approximation from the form given by the diffusion approximation. For simplicity, we shall restrict our attention only to the case where the medium is sufficiently thick to electron scattering.

4.1. Frequency-Dependent Form of Radiative Viscosity

The left-hand side of the transfer equation (10) is written in the inertial frame as

$$\nu\left(\frac{\partial}{\partial t}+\boldsymbol{l}\cdot\boldsymbol{\nabla}\right)n(\nu,\boldsymbol{l})\,, \tag{21}$$

where $n(\nu,\boldsymbol{l})$ is the occupation number of the photon in the inertial frame. From the invariant nature of the photon's occupation number and from the Lorentz transformations (14), we can rewrite expression (21) as

$$\bar{\nu}\left[(1+\boldsymbol{v}\cdot\bar{\boldsymbol{l}})\frac{\partial}{\partial t}+(\bar{\boldsymbol{l}}+\boldsymbol{v})\cdot\boldsymbol{\nabla}\right]_{\nu,\boldsymbol{l}}n(\bar{\nu},\bar{\boldsymbol{l}})\,, \tag{22}$$

where the differentiations, $\partial/\partial t$ and $\boldsymbol{\nabla}$, are operated with fixed ν and $\boldsymbol{l}$.

Since

$$\left.\frac{\partial}{\partial t}\right|_{\nu,\boldsymbol{l}}=\left.\frac{\partial}{\partial t}\right|_{\bar{\nu},\bar{\boldsymbol{l}}}-\bar{\boldsymbol{l}}\cdot\frac{\partial\boldsymbol{v}}{\partial t}\bar{\nu}\frac{\partial}{\partial\bar{\nu}}+\left[-\frac{\partial\boldsymbol{v}}{\partial t}+\bar{\boldsymbol{l}}\left(\bar{\boldsymbol{l}}\cdot\frac{\partial\boldsymbol{v}}{\partial t}\right)\right]\cdot\frac{\partial}{\partial\bar{\boldsymbol{l}}}\,, \tag{23a}$$

$$\boldsymbol{\nabla}|_{\nu,\boldsymbol{l}}=\boldsymbol{\nabla}|_{\bar{\nu},\bar{\boldsymbol{l}}}-(\bar{l}^j\partial_i v_j)\bar{\nu}\frac{\partial}{\partial\bar{\nu}}+(-\partial_i v^k+\bar{l}^j\bar{l}^k\partial_i v_j)\frac{\partial}{\partial\bar{l}^k}\,, \tag{23b}$$

the terms in the brackets of expression (22) can be rewritten by the quantities in the fluid frame as

$$\left[(1+\boldsymbol{v}\cdot\bar{\boldsymbol{l}})\frac{\partial}{\partial t}+(\bar{\boldsymbol{l}}+\boldsymbol{v})\cdot\boldsymbol{\nabla}\right]_{\bar{\nu},\bar{\boldsymbol{l}}}-\left[\bar{\boldsymbol{l}}\cdot\frac{\partial\boldsymbol{v}}{\partial t}+\bar{\boldsymbol{l}}(\bar{\boldsymbol{l}}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right]\bar{\nu}\frac{\partial}{\partial\bar{\nu}}$$
$$+\left\{-\frac{\partial\boldsymbol{v}}{\partial t}-(\bar{\boldsymbol{l}}\cdot\boldsymbol{\nabla})\boldsymbol{v}+\bar{\boldsymbol{l}}\left[\bar{\boldsymbol{l}}\cdot\frac{\partial\boldsymbol{v}}{\partial t}+\bar{\boldsymbol{l}}(\bar{\boldsymbol{l}}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right]\right\}\frac{\partial}{\partial\bar{\boldsymbol{l}}}\,. \tag{24}$$

Substitution of expressions (22) and (24) into the left-hand side of equation (10) gives finally the transfer equation expressed explicitly by the quantities in the fluid frame. This transfer equation in the fluid frame is solved in the remaining of this section.

As mentioned before, we shall restrict our attention to the case where the medium is sufficiently thick to electron scattering. Thus $n(\bar{\nu},\bar{\boldsymbol{l}})$ can be expanded in the inverse powers of $n_e\sigma_T$ as

$$n(\bar{\nu},\bar{\boldsymbol{l}})=n_0(\bar{\nu},\bar{\boldsymbol{l}})+\frac{1}{n_e\sigma_T}n_1(\bar{\nu},\bar{\boldsymbol{l}})+\cdots\,. \tag{25}$$

Note that the subscript 0 on n should not be confused with that used to denote quantities in the electron rest frame.

It is reasonable to assume that the photon occupation number $n_0(\bar{\nu},\bar{\boldsymbol{l}})$ is a local quantity and independent of the direction, for there is no particular preferred direction in the medium when the optical depth is infinite. That is,

$$n_0(\bar{\nu},\bar{\boldsymbol{l}}\,;t,\boldsymbol{r})=n_0[\bar{\nu};T(t,\boldsymbol{r})]\,. \tag{26}$$

In this case, the expansion of equation (10) by the parameter $1/n_e\sigma_T$ shows that as the relation in the lowest order of approximation we have an equation to be satisfied by n_0: the time independent Kompaneets (1957) equation. Without losing generality, it is expressed as

$$T\frac{\partial n_0}{\partial\bar{\nu}}+n_0+n_0^2=0\,. \tag{27}$$

Proceeding to the next order of approximation, we collect terms of the order of $(n_e\sigma_T)^0$ in equation (10) with the help of equations (24) and (25). After performing integrations over the direction, we have

$$\left[1-\frac{2\bar{\nu}}{m_e}-\frac{2}{m_e}\frac{\partial}{\partial\bar{\nu}}(\bar{\nu}^2n_0)\right]n_1(\bar{\nu},\bar{\boldsymbol{l}})$$
$$=\frac{3}{4}\left(1-\frac{2\bar{\nu}}{m_e}+\frac{2T}{m_e}+\mathscr{L}\right)\bar{n}_1(\bar{\nu})+\frac{3}{4}\left(\frac{2\bar{\nu}}{m_e}-\frac{4T}{m_e}-\mathscr{L}\right)\bar{l}^i\bar{f}_1{}^i(\bar{\nu})$$
$$+\frac{3}{4}\left(1-\frac{2\bar{\nu}}{m_e}-\frac{6T}{m_e}+\mathscr{L}\right)\bar{l}^i\bar{l}^j\bar{p}_1{}^{ij}(\bar{\nu})+\frac{3}{4}\left(\frac{2\bar{\nu}}{m_e}+\frac{4T}{m_e}-\mathscr{L}\right)\bar{l}^i\bar{l}^j\bar{l}^k\bar{q}_1{}^{ijk}(\bar{\nu})$$
$$-\frac{dn_0}{dt}\,, \tag{28}$$

where the operators, d/dt and $\mathscr{L}$, are defined as

$$\frac{d}{dt}\equiv\left[(1+\boldsymbol{v}\cdot\bar{\boldsymbol{l}})\frac{\partial}{\partial t}+(\bar{\boldsymbol{l}}+\boldsymbol{v})\cdot\boldsymbol{\nabla}\right]_{\bar{\nu},\bar{\boldsymbol{l}}}-\left[\bar{\boldsymbol{l}}\cdot\frac{\partial\boldsymbol{v}}{\partial t}+\bar{\boldsymbol{l}}(\bar{\boldsymbol{l}}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right]\bar{\nu}\frac{\partial}{\partial\bar{\nu}}\,, \tag{29}$$

$$\mathscr{L}\equiv\frac{1}{m_e}\left(\frac{T}{\bar{\nu}^2}\frac{\partial}{\partial\bar{\nu}}\bar{\nu}^4\frac{\partial}{\partial\bar{\nu}}+\frac{1}{\bar{\nu}^2}\frac{\partial}{\partial\bar{\nu}}\bar{\nu}^4+2n_0\frac{\partial}{\partial\bar{\nu}}\bar{\nu}^2\right), \tag{30}$$

and $\bar{n}_1(\bar{\nu})$, $\bar{f}_1{}^i(\bar{\nu})$, $\bar{p}_1{}^{ij}(\bar{\nu})$, and $\bar{q}_1{}^{ijk}(\bar{\nu})$ are respectively the zeroth, first, second, and third moments of $n_1(\bar{\nu}, \bar{l})$ and defined like equation (12).

Taking the zeroth and the second moments of equation (28) and combining them, we finally have after some manipulations

$$\bar{p}^{ij}(\bar{\nu})=\frac{\delta^{ij}}{3}\bar{n}(\bar{\nu})-\eta(\bar{\nu})\left(\partial_i v^j+\partial_j v^i-\frac{2}{3}\delta^{ij}\partial_k v^k\right), \tag{31}$$

where

$$\begin{aligned}\eta(\bar{\nu})&=-\frac{2}{27n_e\sigma_T}\left[1+\frac{2\bar{\nu}}{m_e}-\frac{2T}{3m_e}+\frac{1}{9}\mathscr{L}+\frac{20}{9m_e}\frac{\partial}{\partial\bar{\nu}}(\bar{\nu}^2n_0)\right]\left(\bar{\nu}\frac{\partial n_0}{\partial\bar{\nu}}\right)\\&=-\frac{2}{27n_e\sigma_T}\left\{1+\frac{T}{9m_e}[-2+17x+(34x-18x^2)n_0-18x^2n_0{}^2]\right\}\left(\bar{\nu}\frac{\partial n_0}{\partial\bar{\nu}}\right),\end{aligned} \tag{32}$$

$$x\equiv\bar{\nu}/T\,. \tag{33}$$

In deriving the second equality of equation (32), we have used equation (27).

In the case of the Thomson scattering in a cold plasma, the terms in the brackets in the first line of equation (32) are unity. That is, the second to the fifth terms in the brackets represent corrections due to the Compton scattering in an electron gas with a velocity dispersion. The correction factor $2\bar{\nu}/m_e$ is due to the fact that the effective cross section of scattering decreases with increasing energy by the Klein–Nishina formula. The third term represents the effects of the presence of the velocity dispersion in the hot plasma. The last two terms among the four corrections can be regarded as representing the effects of the Comptonization processes.

4.2 *Frequency-Integrated Form of Radiative Viscosity*

Integration of equation (32) over the frequency after multiplying equation (32) by $8\pi\bar{\nu}^3$ gives the coefficient of radiative viscosity η. Performing the integration by parts, we have

$$\eta\equiv\int_0^\infty d\bar{\nu}8\pi\bar{\nu}^3\eta(\bar{\nu})=\frac{8E_0}{27n_e\sigma_T}\left(1+\frac{1}{2T}\langle\bar{\nu}^2\rangle_{E_0}-\frac{7}{9}T\right), \tag{34}$$

where

$$\bar{E}_0\equiv\int_0^\infty d\bar{\nu}8\pi\bar{\nu}^3n_0(\bar{\nu}) \tag{35}$$

is the radiation energy density in the fluid frame and

$$\langle\bar{\nu}^2\rangle_{E_0}\equiv\int d\bar{\nu}8\pi\bar{\nu}^5n_0/\bar{E}_0 \tag{36}$$

are the energy-weighted mean values of some quantities. A general form of radiative viscosity is given by Masaki (1981).

When the solution $n_0(\bar{\nu})$ of the Kompaneets (1957) equation (27) is known, we can have the explicit form of radiative viscosity from equation (32) or (34).

4.3. An Example

A typical example is the case where the Planck distribution is taken as $n_0(\bar{\nu})$. That is,

$$n_0(\bar{\nu})=n_{\mathrm{p}}\equiv(e^x-1)^{-1}\,. \tag{37}$$

Of course, the Planck distribution satisfies the Kompaneets (1957) equation (27). The coefficient of radiative viscosity (34) becomes

$$\eta=\frac{8E_{\mathrm{p}}}{27n_{\mathrm{e}}\sigma_{\mathrm{T}}}\left(1+C\frac{T}{m_{\mathrm{e}}}\right)\,. \tag{38}$$

Here $E_{\mathrm{p}}(=aT^4)$ is the radiation energy density for the Planck distribution and C is the numerical factor

$$C=\frac{20}{21}\pi^2-\frac{7}{9}\approx 8.62\,. \tag{39}$$

Thus within our approximations, radiative viscosity increases slightly in comparison with that of the Thomson scattering in a cold plasma (Masaki 1971) by a factor $(1+CT/m_{\mathrm{e}})$, as long as the medium is optically thick to electron scattering.

5. Equations for Radiation and Matter

Returning to the diffusion approximation, we shall now summarize in the inertial frame the frequency-integrated forms of the photon's moment equations and the equations governing the behavior of matter.

Integrating equations (18) and (19) over the frequency after multiplying them by $8\pi\nu^3$, we have equations for radiation:

$$\begin{aligned}\frac{\partial E}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{F}=&-\left(1-\frac{28\langle\nu\rangle_F}{5m_{\mathrm{e}}}+\frac{2T}{5m_{\mathrm{e}}}\right)n_{\mathrm{e}}\sigma_{\mathrm{T}}\boldsymbol{v}\cdot\boldsymbol{F}\\&+\frac{n_{\mathrm{e}}\sigma_{\mathrm{T}}}{m_{\mathrm{e}}}(4T-\langle\nu\rangle_E-\langle\nu n\rangle_E)E+\frac{n_{\mathrm{e}}\sigma_{\mathrm{T}}}{m_{\mathrm{e}}}\frac{6}{5}\langle\nu f\rangle_F\boldsymbol{F}\,,\end{aligned} \tag{40}$$

$$\begin{aligned}\frac{\partial F^i}{\partial t}+\partial_j P^{ij}=&-\left(1-\frac{14\langle\nu\rangle_F}{5m_{\mathrm{e}}}+\frac{2T}{5m_{\mathrm{e}}}\right)n_{\mathrm{e}}\sigma_{\mathrm{T}}F^i\\&+\left(1-\frac{7\langle\nu\rangle_E}{2m_{\mathrm{e}}}+\frac{2T}{5m_{\mathrm{e}}}\right)n_{\mathrm{e}}\sigma_{\mathrm{T}}\frac{4}{3}Ev^i\\&-\frac{n_{\mathrm{e}}\sigma_{\mathrm{T}}}{m_{\mathrm{e}}}\left[4T-\langle\nu\rangle_F-2\langle\nu n\rangle_F-7\left\langle\frac{\partial}{\partial\nu}(\nu^2 n)\right\rangle_F\right]\frac{2}{5}F^i\,,\end{aligned} \tag{41}$$

where

$$E\equiv\int d\nu\, 8\pi\nu^3 n(\nu)\,, \tag{42a}$$

$$F^i \equiv \int d\nu 8\pi\nu^3 f^i(\nu) , \tag{42b}$$

$$P^{ij} \equiv \int d\nu 8\pi\nu^3 p^{ij}(\nu) . \tag{42c}$$

Here we have introduced the energy-weighted and the flux-weighted mean quantities defined, for example, as

$$\langle A\rangle_E \equiv \int d\nu 8\pi\nu^3 A(\nu) n(\nu)/E , \tag{43a}$$

$$\langle A\rangle_F \equiv \int d\nu 8\pi\nu^3 A(\nu) \boldsymbol{f}(\nu)/\boldsymbol{F} . \tag{43b}$$

Integrating equation (20), we can express the radiation tensor P^{ij} in equation (41) in terms of E and F^i as

$$P^{ij} = \frac{\delta^{ij}}{3}E + v^i F^j + v^j F^i - \frac{2}{3}\delta^{ij} v_k F^k . \tag{44}$$

The terms proportional to $\langle\nu\rangle_E$ or $\langle\nu\rangle_F$ in the first parentheses on the right-hand side of equation (40) and in the first and the second parentheses on the right-hand side of equation (41) come mainly from the Klein–Nishina reduction of the total cross section and partially from the fact that the probability distribution of scattering angle changes from that of the Thomson scattering. The terms proportional to T in the above-mentioned parentheses represent the effects of velocity dispersion of electrons in the hot plasma, as already mentioned. The second and the third terms on the right-hand side of equation (40) represent the Comptonization effect. That is, if the electron temperature is higher than the photon temperature, the energy flows into the photon gas from the electron gas, and vice versa. The term $(4T - \langle\nu\rangle_E - \langle\nu n\rangle_E)$ vanishes when the radiation field is Planckian. Similarly, the last term on the right-hand side of equation (41) shows the momentum exchange between the photon gas and the electron gas through Comptonization.

Eliminating F^i from equations (40) and (41), we obtain the energy equation for the photon gas (cf. Blandford and Payne 1981) as

$$\begin{aligned}
&\frac{\partial E}{\partial t} + \left[1 - \frac{14}{3m_e}\langle\nu\rangle_E + \frac{76}{15m_e}\langle\nu\rangle_F - \frac{8T}{5m_e} + \frac{4}{5m_e}\langle\nu n\rangle_F + \frac{14}{5m_e}\left\langle\frac{\partial}{\partial\nu}(\nu^2 n)\right\rangle_F\right]\boldsymbol{v}\cdot\boldsymbol{\nabla}E \\
&\quad = \partial_i \chi \partial_j P^{ij} \\
&\qquad - \frac{4}{3}E\boldsymbol{\nabla}\left[1 - \frac{7}{2m_e}\langle\nu\rangle_E + \frac{16}{5m_e}\langle\nu\rangle_F - \frac{8T}{5m_e} + \frac{4}{5m_e}\langle\nu n\rangle_F + \frac{14}{5m_e}\left\langle\frac{\partial}{\partial\nu}(\nu^2 n)\right\rangle_F\right]\boldsymbol{v} \\
&\qquad + \frac{n_e\sigma_T}{m_e}(4T - \langle\nu\rangle_E - \langle\nu n\rangle_E)E \\
&\qquad + \frac{n_e\sigma_T}{m_e}\frac{6}{5}\langle\nu \boldsymbol{f}\rangle_F\left(-\frac{1}{n_e\sigma_T}\partial_j P^{ij} + \frac{4}{3}Ev^i\right) ,
\end{aligned} \tag{45}$$

where χ is the coefficient of radiative diffusion of the photon gas and

$$\chi=\frac{1}{n_e\sigma_T}\left[1+\frac{16}{5m_e}\langle\nu\rangle_F-\frac{2T}{m_e}+\frac{4}{5m_e}\langle\nu n\rangle_F+\frac{14}{5m_e}\left\langle\frac{\partial}{\partial\nu}(\nu^2 n)\right\rangle_F\right]. \tag{46}$$

To complete the system of equations for radiation hydrodynamics, we need equations for matter, which are derived from the conservation of the energy-momentum tensor for the sum of radiation and matter (Hsieh and Spiegel 1976). Within the present approximations, we have the equation of motion:

$$\begin{aligned}
&\rho\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v}+\left(\boldsymbol{v}\frac{\partial}{\partial t}+\boldsymbol{\nabla}\right)p-\rho\boldsymbol{\nabla}\phi\\
&=-\left(\frac{\partial \boldsymbol{F}}{\partial t}+\partial_j P^{ij}\right)+\boldsymbol{v}\left(\frac{\partial E}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{F}\right)\\
&=\left[1-\frac{16}{5m_e}\langle\nu\rangle_F+\frac{2T}{m_e}-\frac{4}{5m_e}\langle\nu n\rangle_F-\frac{14}{5m_e}\left\langle\frac{\partial}{\partial\nu}(\nu^2 n)\right\rangle_F\right]n_e\sigma_T\boldsymbol{F}\\
&\quad-\left(1-\frac{11}{4m_e}\langle\nu\rangle_E-\frac{13T}{5m_e}+\frac{3}{4m_e}\langle\nu n\rangle_E\right)n_e\sigma_T\frac{4}{3}E\boldsymbol{v}\\
&\quad+\frac{n_e\sigma_T}{m_e}\frac{6}{5}(\langle\nu\boldsymbol{f}\rangle_F\cdot\boldsymbol{F})\boldsymbol{v}\,,
\end{aligned}\tag{47}$$

and the energy equation:

$$\begin{aligned}
&\rho\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)e+p\boldsymbol{\nabla}\cdot\boldsymbol{v}\\
&=-\left(\frac{\partial E}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{F}\right)+\boldsymbol{v}\left(\frac{\partial \boldsymbol{F}}{\partial t}+\partial_j P^{ij}\right)\\
&=-\frac{n_e\sigma_T}{m_e}(4T-\langle\nu\rangle_E-\langle\nu n\rangle_E)E-\frac{n_e\sigma_T}{m_e}\frac{6}{5}\langle\nu\boldsymbol{f}\rangle_F\cdot\boldsymbol{F}\\
&\quad-\frac{n_e\sigma_T}{m_e}\left[4T+6\langle\nu\rangle_F-2\langle\nu n\rangle_F-7\left\langle\frac{\partial}{\partial\nu}(\nu^2 n)\right\rangle_F\right]\frac{2}{5}\boldsymbol{v}\cdot\boldsymbol{F}\,.
\end{aligned}\tag{48}$$

Here ρ is the density, p is the pressure, e is the internal energy of the matter, and ϕ is the potential of the external force acting on the matter.

In conclusion, we briefly mention several notable modifications of the equations for matter due to the presence of Compton processes. The first term on the right-hand side of equation (47) represents the radiation force. The coefficient of this force, the terms in the brackets, decreases with increase of the photon energy. If we assume that the first moment, $\boldsymbol{f}(\nu)$, is proportional to the gradient of the zeroth moment, $\boldsymbol{\nabla}n(\nu)$, and that the radiation field is Planckian with temperature T as done by Sampson (1959), then this radiation force is reduced to $(1-4\pi^2T/3m_e)n_e\sigma_T F^i\sim(1-13.2T/m_e)n_e\sigma_T F^i$. This is in good agreement with the result obtained by Sampson (1959). We note that in the same approximation χ becomes $(1+4\pi^2T/3m_e)/n_e\sigma_T$. The radiation drag, the second term on the right-hand side of equation (47), decreases with increase of the photon energy. When the radiation field is Planckian, it is reduced to $\sim(1-13.0T/m_e)4n_e\sigma_T E v^i/3$. The first term on the right-hand side of equation (48) describes the energy exchange between the photon gas and the matter and vanishes for the Planckian radiation field with temperature T. Finally, the third term on the right-hand side of equation (48) represents the work done between radiation and matter. The flux means in this term can be evaluated like the case of the radiation force.

Equations (40), (41), (44), (47), and (48), with the equation of continuity, are thus the basic equations for radiation hydrodynamics obtained in this paper.

6. Discussion

We have examined the radiative transfer and the radiation hydrodynamics in a hot moving plasma under the Compton scattering. Our results are valid only to the first order in $h\nu/m_e c^2$, $kT/m_e c^2$, and v/c. With the increase of photon energy, the total cross section of scattering decreases by the Klein–Nishina formula. In addition, at the scattering there occurs the energy and momentum exchange between the electron and the photon. The amounts of the exchanged energy and momentum depend upon the scattering direction. Furthermore, the fact that the electrons have a velocity dispersion should be taken into consideration. As a result, the radiative viscosity, the radiative diffusivity, the radiation force, the radiation drag, etc., suffer from complicated changes in comparison with the case of the Thomson scattering in the cold plasma.

In this paper radiative viscosity has been examined only in a particular case where the medium is sufficiently thick to electron scattering and hence both the electron temperature and photon temperature are the same. This, however, will not be always the case in practical situations. For instance, in the inner part of accretion disk around a black hole, a high temperature gas may interact with soft photons under an optically thin configuration (e.g., Shapiro et al. 1976). In such a case, the mean momentum exchange at a collision between a photon and an electron will be large because of a large temperature difference between the photon gas and the electron gas. On the other hand, the photon number is small in such disks, since the temperature of the soft photon gas is low. The above two conditions act in opposite directions in the determination of the total amount of the momentum exchange between the photon gas and the electron gas in unit volume. Hence a careful treatment is required to know the magnitude of radiative viscosity in such a multitemperature optically-thin gas.

The examination of the Compton scattering radiation hydrodynamics in the trans-relativistic regime ($h\nu \sim m_e c^2$ or $kT \sim m_e c^2$), which remains in this paper as future work, will be important as well as of interest to practical applications, since the difference between the Compton scattering and the Thomson scattering becomes prominent in such a high energy regime. However, its study in the extreme relativistic regime will be unimportant, because the cross section of the photon-electron interaction becomes sufficiently small in such a regime and other processes such as double Compton scattering process and a e^+e^- pair production may dominate the Compton processes. The cross section of the double (radiative) Compton scattering is of the order of $\alpha = 1/137$ in comparison with that of the Thomson scattering. It is, however, important for $kT \gtrsim 0.1 m_e c^2$ (Weaver 1976) because of the Klein–Nishina reduction of the single Compton scattering. Furthermore, the double Compton scattering is of importance as a photon source (Lightman 1981; Thorne 1981). Pair production becomes dominant at temperatures somewhat above $m_e c^2$. Formulation of equations for radiative transfer and for radiation hydrodynamics including the above processes is left open for the future.

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