On the Pinsky Phenomenon of Fourier Series of the Indicator Function in Several Variables

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Let $n \geq 3$. We point out that, for the indicator function of the n-dimensional ball, the spherical partial sum of Fourier series oscillates at the center of the ball, and that, if the radius of the ball is $1/k$ ($k = 2, 3, \cdots$), then the period of the oscillation is k. Moreover, we give graphs of the partial sums when $n = 3, 4, 5$ and 6. We can see the Pinsky phenomenon and the Gibbs phenomenon in them. In the cases of $n = 5$ and 6 we can see another phenomenon.

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I. Introduction

In one dimension, the behavior of Fourier series at a point depends only on the behavior of the function in a neighborhood of that point. In particular, if the function is zero on an interval, then the Fourier series converges to zero on that interval. However, in two or more dimensions, this localization property does not generally hold (see for example [1, 2, 14]).

In two or more dimensions, there are many ways to add up the terms of Fourier series; spherical partial sum, rectangular partial sum, square partial sum, etc. See [3, 4, 6] for the rectangular or square partial sum.

In this paper we consider the spherical partial sum. Pinsky, Stanton and Trapa [13] and Kuratsubo [8, 9] investigated the convergence of the spherical partial sums of radial functions. Pinsky, Stanton and Trapa [13] showed, for example, that the spherical partial sum of Fourier series of the indicator function of the n-dimensional ball diverges at the center of the ball when $n \geq 3$.

In this paper, we point out that, when $n \geq 3$, for the indicator function of the n-dimensional ball, the spherical partial sum of Fourier series oscillates at the center of the ball, and that, if the radius of the ball is $1/k$ ($k = 2, 3, \cdots$), then the period of the oscillation is k. Moreover, we give graphs of the partial sums when $n = 3, 4, 5$ and 6. We can see the Pinsky phenomenon and the Gibbs phenomenon in them. In the cases of $n = 5$ and 6 we can see another phenomenon.

The Fourier coefficients of a function F on $\mathbb{T}^n$ and its spherical partial sum are defined by

$$\hat{F}(m) = \int_{\mathbb{T}^n} F(x) e^{-2\pi i m x}\, dx, \quad m = (m_1, \cdots, m_n) \in \mathbb{Z}^n,$$

$$S_\lambda(F : x) = \sum_{|m| \leq \lambda} \hat{F}(m) e^{2\pi i m x}, \quad |m| = \sqrt{\sum_{k=1}^{n} {m_k}^2}, \quad x \in \mathbb{T}^n,$$

respectively, where $\mathbb{Z}^n$ denotes the n-dimensional integer lattice, $\mathbb{T}^n$ denotes the n-dimensional torus, whose points are written $(x_1, \cdots, x_n)$, $-\frac{1}{2} \leq x_k \leq \frac{1}{2}$, and, the inner product mx denotes $\sum_{k=1}^n m_k x_k$.

Let χ_a be the indicator function of the ball centered at 0 and of radius $a > 0$, i.e.

$$\chi_a(x) = \begin{cases} 1 & |x| \leq a, \\ 0 & |x| > a. \end{cases}$$

Let

$$\overline{\chi_a}(x) = \begin{cases} 1 & |x| < a, \\ \frac{1}{2} & |x| = a, \\ 0 & |x| > a, \end{cases}$$

and

$$F_a(x) = \sum_{m \in \mathbb{Z}^n} \chi_a(x + m), \quad \overline{F_a}(x) = \sum_{m \in \mathbb{Z}^n} \overline{\chi_a}(x + m).$$

If $0 < a < \frac{1}{2}$, then F_a is equal on $\mathbb{T}^n$ to the indicator function of the ball with radius a and centered at 0.

Our main results are the following.

Theorem 1.1. *Let $n \geq 3$ and $a > 0$. Then $S_\lambda(F_a, 0)$ oscillates as $\lambda = 1, 2, 3, \cdots$ and $\lambda \to \infty$. If $a = 1/k$ ($k = 2, 3, \cdots$), then the period of the oscillation is k. More precisely,*

$$S_\lambda(F_a : 0) = \overline{F_a}(0) - \frac{\pi^{\frac{n}{2}-2}}{\Gamma(\frac{n}{2})} (a\lambda)^{\frac{n-3}{2}} \sin\left(2\pi a\lambda - \frac{n-3}{4}\pi\right) + O\left(\lambda^{\max\left(0, \frac{n-5}{2}\right)}\right).$$

Particularly, in the cases $n = 3, 4, 5$, we have the following.

Theorem 1.2. *Let $a > 0$.*

(1) *If $n = 3$, then*

$$S_\lambda(F_a : 0) = \overline{F_a}(0) - \frac{2}{\pi}\sin(2\pi a\lambda) + o(1).$$

(2) *If $n = 4$, then*

$$S_\lambda(F_a : 0) = \overline{F_a}(0) - (a\lambda)^{\frac{1}{2}}\sin\left(2\pi a\lambda - \frac{1}{4}\pi\right) + o(1).$$

(3) *If $n = 5$, then*

$$S_\lambda(F_a : 0) = \overline{F_a}(0) + \frac{4}{3}a\lambda\cos(2\pi a\lambda) - \frac{8}{3\pi}\sin(2\pi a\lambda) + O(1).$$

Example 1.1. Let $n = 3$ and $a = 1/4$. Then

$$S_\lambda(F_{1/4}, 0) = 1 - \frac{2}{\pi}\sin\frac{\pi}{2}\lambda + o(1) = \begin{cases} 1 + o(1) & \lambda \equiv 0 \bmod 4, \\ 1 - \dfrac{2}{\pi} + o(1) & \lambda \equiv 1 \bmod 4, \\ 1 + o(1) & \lambda \equiv 2 \bmod 4, \\ 1 + \dfrac{2}{\pi} + o(1) & \lambda \equiv 3 \bmod 4. \end{cases}$$

Example 1.2. Let $n = 4$ and $a = 1/4$. Then

$$S_\lambda(F_{1/4}, 0) = 1 - \sqrt{\frac{\lambda}{4}}\sin\left(\frac{\pi}{2}\lambda - \frac{\pi}{4}\right) + o(1) = \begin{cases} 1 + \sqrt{\dfrac{\lambda}{8}} + o(1) & \lambda \equiv 0 \bmod 4, \\ 1 - \sqrt{\dfrac{\lambda}{8}} + o(1) & \lambda \equiv 1 \bmod 4, \\ 1 - \sqrt{\dfrac{\lambda}{8}} + o(1) & \lambda \equiv 2 \bmod 4, \\ 1 + \sqrt{\dfrac{\lambda}{8}} + o(1) & \lambda \equiv 3 \bmod 4. \end{cases}$$

Example 1.3. Let $n = 5$ and $a = 1/4$. Then

$$S_\lambda(F_{1/4}, 0) = 1 + \frac{\lambda}{3}\cos\frac{\pi}{2}\lambda - \frac{8}{3\pi}\sin\frac{\pi}{2}\lambda + O(1)$$

$$= \begin{cases} 1 + \dfrac{\lambda}{3} + O(1) & \lambda \equiv 0 \bmod 4, \\ 1 - \dfrac{8}{3\pi} + O(1) & \lambda \equiv 1 \bmod 4, \\ 1 - \dfrac{\lambda}{3} + O(1) & \lambda \equiv 2 \bmod 4, \\ 1 + \dfrac{8}{3\pi} + O(1) & \lambda \equiv 3 \bmod 4. \end{cases}$$

We state known results in the next section and prove our main results in the third section. We give graphs of $S_\lambda(F_{1/4})$ for $n = 3, 4, 5, 6$ in Section 4. For Example 1.1 see Figure 1, for Example 1.2 see Figure 2, and, for Example 1.3 see Figures 3 and 4.

II. Known results

In this section we state known results to compare with our main results. We use Lemmas 2.5 and 2.6 to prove our main results in the third section.

2.1. **Fourier transform of radial functions.** The Fourier transform of a function f on $\mathbb{R}^n$ and its spherical partial sum are defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi x}\,dx, \quad \xi \in \mathbb{R}^n,$$

$$f_\lambda(x) = \int_{|\xi|<\lambda} \hat{f}(\xi)e^{2\pi i\xi x}\,d\xi, \quad |\xi| = \sqrt{\sum_{k=1}^{n} {\xi_k}^2}, \quad x \in \mathbb{R}^n,$$

respectively, where $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, whose points are written $x = (x_1, \cdots, x_n)$, $-\infty < x_k < \infty$, the inner product ξx denotes $\sum_{k=1}^n \xi_k x_k$ for $\xi, x \in \mathbb{R}^n$.

For the indicator function of the ball

$$\chi_a(x) = \begin{cases} 1 & |x| \le a, \\ 0 & |x| > a, \end{cases}$$

it is known that the Fourier transform of χ_a is the following:

$$\hat{\chi}_a(\xi) = \begin{cases} \dfrac{\pi^{\frac{n}{2}} a^n}{\Gamma(\frac{n}{2}+1)} & \text{for } \xi = 0, \\[2ex] a^{\frac{n}{2}} \dfrac{J_{\frac{n}{2}}(2\pi|\xi|a)}{|\xi|^{\frac{n}{2}}} & \text{for } \xi \ne 0, \end{cases} \tag{2.1}$$

where J_α is the Bessel function of the first kind of order α. Its spherical partial sum denoted by $\chi_{a,\lambda}(x)$, i.e.

$$\chi_{a,\lambda}(x) = \int_{|\xi|\le\lambda} \hat{\chi}_a(\xi)e^{2\pi i\xi x}\,d\xi.$$

There are several results for the spherical partial sum of the Fourier transform of the radial functions (see [5, 8, 10, 11, 13], etc.). For the indicator function of the ball, the following is known.

Theorem 2.1. *Let $x = 0$.*

(1) *If $n = 1$ or $n = 2$, then*

$$\lim_{\lambda\to\infty} \chi_{a,\lambda}(0) = \overline{\chi_a}(0) = 1.$$

(2) *If $n = 3$, then*

$$\liminf_{\lambda\to\infty} \chi_{a,\lambda}(0) = 1 - \frac{2}{\pi}, \quad \limsup_{\lambda\to\infty} \chi_{a,\lambda}(0) = 1 + \frac{2}{\pi}.$$

(3) *If $n \geq 4$, then*

$$\liminf_{\lambda\to\infty} \lambda^{\frac{3-n}{2}} \chi_{a,\lambda}(0) = -\frac{\pi^{\frac{n-4}{2}} a^{\frac{n-3}{2}}}{\Gamma(\frac{n}{2})}, \quad \limsup_{\lambda\to\infty} \lambda^{\frac{3-n}{2}} \chi_{a,\lambda}(0) = \frac{\pi^{\frac{n-4}{2}} a^{\frac{n-3}{2}}}{\Gamma(\frac{n}{2})}.$$

Theorem 2.2. *Let $x \neq 0$. Then, for all n,*

$$\lim_{\lambda\to\infty} \chi_{a,\lambda}(x) = \overline{\chi_a}(x).$$

2.2. **Fourier series of radial functions.** There are several results for the spherical partial sum of Fourier series of the radial functions (see [5, 7, 8, 9, 12, 13], etc.). For the indicator function of the ball, the following is known.

Theorem 2.3 ([12, 13])**.** *If $n \geq 3$, then the spherical partial sum of the n-dimensional Fourier series of the indicator function of the ball $|x| \leq a$ diverges at the center $x = 0$.*

We denote by $\mathbb{Q}^n$ the set of all rational points in $\mathbb{R}^n$.

Theorem 2.4 ([8], p. 202)**.** *For $x \neq 0$ we have the following:*

(1) *If $2 \leq n \leq 4$,*

$$\lim_{\lambda\to\infty} S_\lambda\left(F_a : x\right) = \overline{F_a}(x).$$

(2) *If $n \geq 5$ and $x \notin \mathbb{Q}^n$,*

$$\limsup_{\lambda\to\infty} \left(\lambda^{\frac{5-n}{2}} \left|S_\lambda\left(F_a : x\right) - \overline{F_a}(x)\right|\right) = 0.$$

(3) *If $n \geq 5$ and $x \in \mathbb{Q}^n$,*

$$\limsup_{\lambda\to\infty} \left(\lambda^{\frac{5-n}{2}} \left|S_\lambda\left(F_a : x\right) - \overline{F_a}(x)\right|\right) < +\infty.$$

(4) *If $n \geq 5$,*

$$\lim_{\lambda\to\infty} S_\lambda\left(F_a : x\right) = \overline{F_a}(x) \quad \textit{for almost all } x.$$

From (2.1) it follows that, for $a > 0$,

$$\hat{F}_a(m) = \hat{\chi_a}(m) = \begin{cases} \dfrac{\pi^{\frac{n}{2}} a^n}{\Gamma(\frac{n}{2}+1)} & \text{for } m = 0, \\ a^{\frac{n}{2}} \dfrac{J_{\frac{n}{2}}\left(2\pi|m|a\right)}{|m|^{\frac{n}{2}}} & \text{for } m \neq 0. \end{cases} \tag{2.2}$$

Lemma 2.5 ([8], p. 204)**.** *Let $a > 0$. Then*

$$\begin{aligned} S_\lambda(F_a : x) &= \sum_{|m|<\lambda} \hat{\chi_a}(m) e^{2\pi i m x} \\ &= \overline{F_a}(x) + (\chi_{a,\lambda}(0) - 1)\delta(x) + a^{\frac{n}{2}} \sum_{l=0}^{k_0} (\pi a)^l P_l(\lambda^2 : x)\Lambda_l(a, \lambda^2) + O(\lambda^{-1}), \end{aligned}$$

where

$$\delta(x) = \begin{cases} 1 & x \in \mathbb{Z}^n, \\ 0 & x \notin \mathbb{Z}^n, \end{cases} \qquad k_0 = \min\left\{k \in \mathbb{Z} : \frac{n-1}{2} < k\right\},$$

$$P_l(t : x) = \frac{1}{\Gamma(l+1)} \sum_{|m|^2 < t} (t - |m|^2)^l e^{2\pi i m x} - \frac{\pi^{\frac{n}{2}} t^{\frac{n}{2}+l}}{\Gamma(\frac{n}{2}+l+1)} \delta(x),$$

$$\Lambda_l(a, s) = \frac{J_{\frac{n}{2}+l}(2\pi a s^{\frac{1}{2}})}{s^{\frac{1}{2}(\frac{n}{2}+l)}}.$$

Lemma 2.6 ([8], p. 207)**.** (1) *If* $2 \le n \le 4$, *then*

$$P_l(\lambda^2 : x)\Lambda_l(a, \lambda^2) = o(1) \quad \textit{for} \quad 0 \le l \le k_0.$$

(2) *If* $n \ge 5$ *and* $x \in \mathbb{Q}^n$, *then*

$$P_l(\lambda^2 : x)\Lambda_l(a, \lambda^2) = O(\lambda^{\frac{n-5}{2}}) \quad \textit{for} \quad 0 \le l \le k_0.$$

(3) *If* $n \ge 5$ *and* $x \notin \mathbb{Q}^n$, *then*

$$P_l(\lambda^2 : x)\Lambda_l(a, \lambda^2) = o(\lambda^{\frac{n-5}{2}}) \quad \textit{for} \quad 0 \le l \le k_0.$$

III. Proof of main results

We use a formula of the Bessel function:

$$\begin{aligned} J_\alpha(s) &= \sqrt{\frac{2}{\pi s}} \cos\left(s - \frac{2\alpha+1}{4}\pi\right) + O(s^{-\frac{3}{2}}) \\ &= \sqrt{\frac{2}{\pi s}} \sin\left(s - \frac{2\alpha-1}{4}\pi\right) + O(s^{-\frac{3}{2}}) \quad \text{as} \quad s \to \infty. \end{aligned} \tag{3.1}$$

For $\alpha = 3/2,\ 5/2$, we have

$$J_{\frac{3}{2}}(s) = \sqrt{\frac{2}{\pi s}} \left(-\cos s + \frac{\sin s}{s}\right), \tag{3.2}$$

$$J_{\frac{5}{2}}(s) = \sqrt{\frac{2}{\pi s}} \left(-\sin s - \frac{3}{s}\cos s + \frac{3}{s^2}\sin s\right). \tag{3.3}$$

Let

$$G_\alpha(t) = \int_0^t s^{\alpha-1} J_\alpha(s)\, ds.$$

Then, from (2.1) it follows that

$$\chi_{a,\lambda}(0) = \frac{1}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} \int_0^{2\pi a\lambda} s^{\frac{n}{2}-1} J_{\frac{n}{2}}(s)\, ds = \frac{1}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} G_{\frac{n}{2}}(2\pi a\lambda). \tag{3.4}$$

In the following, Subsections 3.1–3.3 are to prove Theorem 1.2. By Subsection 3.4 and Theorem 1.2, we have Theorem 1.1.

3.1. **The case of $n = 3$.** Using (3.2) and (3.4), we have

$$G_{\frac{3}{2}}(t) = \sqrt{\frac{2}{\pi}} \int_0^t \left(-\cos s + \frac{\sin s}{s} \right) ds = -\sqrt{\frac{2}{\pi}} \sin t + \sqrt{\frac{2}{\pi}} \int_0^t \frac{\sin s}{s}\, ds,$$

and

$$\begin{aligned} \chi_{a,\lambda}(0) &= \sqrt{\frac{2}{\pi}} G_{\frac{3}{2}}(2\pi a\lambda) \\ &= -\frac{2}{\pi} \sin(2\pi a\lambda) + \frac{2}{\pi} \int_0^{2\pi a\lambda} \frac{\sin s}{s}\, ds \\ &= -\frac{2}{\pi} \sin(2\pi a\lambda) + 1 + o(1). \end{aligned}$$

By Lemmas 2.5 and 2.6, we have

$$S_\lambda(F_a : 0) = \overline{F_a}(0) - \frac{2}{\pi} \sin(2\pi a\lambda) + o(1).$$

3.2. **The case of $n = 4$.** We note that

$$\frac{d}{ds}\left(\frac{J_1(s)}{s} \right) = -\frac{J_2(s)}{s} \quad \text{and} \quad \int_0^t J_1(s)\, ds = -J_0(t) + 1.$$

By the integration by parts, we have

$$\begin{aligned} G_2(t) =& \int_0^t sJ_2(s)\, ds = -\int_0^t s^2 \left(-\frac{J_2(s)}{s} \right) ds \\ =& -\left[s^2 \left(\frac{J_1(s)}{s} \right) \right]_0^t + \int_0^t 2s \left(\frac{J_1(s)}{s} \right) ds \\ =& -tJ_1(t) - 2J_0(t) + 2. \end{aligned}$$

Using (3.1) and (3.4), we have

$$G_2(t) = -\sqrt{\frac{2}{\pi}}\, t^{\frac{1}{2}} \sin\left(t - \frac{1}{4}\pi \right) + 2 + O(t^{-\frac{1}{2}}),$$

and

$$\begin{aligned} \chi_{a,\lambda}(0) &= \frac{1}{2} G_2(2\pi a\lambda) \\ &= -\frac{1}{2}\sqrt{\frac{2}{\pi}}\, (2\pi a\lambda)^{\frac{1}{2}} \sin\left(2\pi a\lambda - \frac{1}{4}\pi \right) + 1 + O(\lambda^{-\frac{1}{2}}) \\ &= -(a\lambda)^{\frac{1}{2}} \sin\left(2\pi a\lambda - \frac{1}{4}\pi \right) + 1 + O(\lambda^{-\frac{1}{2}}). \end{aligned}$$

By Lemmas 2.5 and 2.6, we have

$$S_\lambda(F_a : 0) = \overline{F_a}(0) - (a\lambda)^{\frac{1}{2}} \sin\left(2\pi a\lambda - \frac{1}{4}\pi \right) + o(1).$$

3.3. **The case of $n = 5$.** Using (3.3) and (3.4), we have

$$\begin{aligned} G_{\frac{5}{2}}(t) &= \sqrt{\frac{2}{\pi}} \int_0^t \left(-s \sin s - 3 \cos s + 3 \frac{\sin s}{s} \right) ds \\ &= \sqrt{\frac{2}{\pi}}\, t \cos t - 4 \sqrt{\frac{2}{\pi}} \sin t + 3 \sqrt{\frac{2}{\pi}} \int_0^t \frac{\sin s}{s}\, ds, \end{aligned}$$

and

$$\begin{aligned} \chi_{a,\lambda}(0) &= \frac{1}{3} \sqrt{\frac{2}{\pi}} G_{\frac{5}{2}}(2\pi a\lambda) \\ &= \frac{2}{3\pi}(2\pi a\lambda)\cos(2\pi a\lambda) - \frac{8}{3\pi}\sin(2\pi a\lambda) + \frac{2}{\pi}\int_0^{2\pi a\lambda} \frac{\sin s}{s}\, ds \\ &= \frac{4}{3} a\lambda \cos(2\pi a\lambda) - \frac{8}{3\pi}\sin(2\pi a\lambda) + 1 + o(1). \end{aligned}$$

By Lemmas 2.5 and 2.6, we have

$$S_\lambda(F_a : 0) = \overline{F_a}(0) + \frac{4}{3} a\lambda \cos(2\pi a\lambda) - \frac{8}{3\pi}\sin(2\pi a\lambda) + O(1).$$

3.4. **The case of $n \geq 6$.** We note that

$$\frac{d}{ds}\left(\frac{J_{\alpha-1}(s)}{s^{\alpha-1}} \right) = -\frac{J_\alpha(s)}{s^{\alpha-1}}.$$

By the integration by parts, we have

$$\begin{aligned} G_\alpha(t) &= -\int_0^t s^{2\alpha-2} \left(-\frac{J_\alpha(s)}{s^{\alpha-1}} \right) ds \qquad (3.5) \\ &= -t^{2\alpha-2}\left(\frac{J_{\alpha-1}(t)}{t^{\alpha-1}} \right) + (2\alpha-2)\int_0^t s^{2\alpha-3}\left(\frac{J_{\alpha-1}(s)}{s^{\alpha-1}} \right) ds \\ &= -t^{\alpha-1} J_{\alpha-1}(t) + 2(\alpha-1) G_{\alpha-1}(t). \end{aligned}$$

By (3.1) we have $t^{\alpha-1}J_{\alpha-1}(t) = O(t^{\alpha-\frac{3}{2}})$. We have already that $G_2(t) = O(t^{\frac{1}{2}})$ in the case of $n = 4$ and $G_{\frac{5}{2}}(t) = O(t)$ in the case of $n = 5$. Then we have by induction $G_\alpha(t) = O(t^{\alpha-\frac{3}{2}})$ for $\alpha = n/2$, $n = 4, 5, 6, \cdots$. Using (3.5) and (3.1) again, we have

$$G_\alpha(t) = -\sqrt{\frac{2}{\pi}}\, t^{\alpha-\frac{3}{2}} \sin\left(t - \frac{2\alpha-3}{4}\pi \right) + O(t^{\alpha-1-\frac{3}{2}}),$$

for $\alpha = n/2$, $n = 6, 7, 8, \cdots$. Using (3.4), we have

$$\begin{aligned} \chi_{a,\lambda}(0) &= \frac{1}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} G_\alpha(2\pi a\lambda) \\ &= -\frac{1}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} \sqrt{\frac{2}{\pi}}\, (2\pi a\lambda)^{\frac{n-3}{2}} \sin\left(2\pi a\lambda - \frac{n-3}{4}\pi \right) + O(\lambda^{\frac{n-5}{2}}) \\ &= -\frac{\pi^{\frac{n}{2}-2}}{\Gamma(\frac{n}{2})} (a\lambda)^{\frac{n-3}{2}} \sin\left(2\pi a\lambda - \frac{n-3}{4}\pi \right) + O(\lambda^{\frac{n-5}{2}}). \end{aligned}$$

By Lemmas 2.5 and 2.6, we have

$$S_\lambda(F_a:0)=\overline{F_a}(0)-\frac{\pi^{\frac{n}{2}-2}}{\Gamma(\frac{n}{2})}(a\lambda)^{\frac{n-3}{2}}\sin\left(2\pi a\lambda-\frac{n-3}{4}\pi\right)+O(\lambda^{\frac{n-5}{2}}).$$

IV. Graphs

In this section, we give graphs of the spherical partial sums of Fourier series for the indicator function of the 3–6 dimensional balls of radius $a=1/4$. We use Mathematica.

To express the graphs of $S_\lambda(F_{1/4})$ in two dimensions, we calculate

$$S_\lambda(F_{1/4}) \quad \text{for} \quad (x_1,x_2,\cdots,x_n)=(x,0,\cdots,0), \quad -\frac{1}{2}<x<\frac{1}{2}.$$

Let

$$U(k)=\begin{cases}\dfrac{\pi^{\frac{n}{2}}a^n}{\Gamma(\frac{n}{2}+1)} & \text{for } k=0,\\[2ex] a^{\frac{n}{2}}\dfrac{J_{\frac{n}{2}}\left(2\pi k^{\frac{1}{2}}a\right)}{k^{\frac{n}{4}}} & \text{for } k=1,2,3,\cdots.\end{cases}$$

Then, by (2.2), we have $U(|m|^2)=\hat{F}_a(m)$ and

$$S_\lambda(F_a:x_1,x_2,\cdots,x_n)=\sum_{{m_1}^2+{m_2}^2+\cdots+{m_n}^2\le\lambda^2}U(|m|^2)e^{2\pi i(m_1x_1+m_2x_2+\cdots+m_nx_n)}$$
$$\text{for} \quad m=(m_1,m_2,\cdots,m_n)\in\mathbb{Z}^n.$$

Then we have

$$\begin{aligned}S_\lambda(F_a:x,0,\cdots,0)&=\sum_{{m_1}^2+{m_2}^2+\cdots+{m_n}^2\le\lambda^2}U(|m|^2)e^{2\pi i m_1x}\\&=\sum_{{m_1}^2+{m_2}^2+\cdots+{m_n}^2\le\lambda^2}U(|m|^2)\cos 2\pi m_1x\\&=\sum_{m_1=-\lambda}^{\lambda}\left(\sum_{{m_2}^2+\cdots+{m_n}^2\le\lambda^2-{m_1}^2}U(|m|^2)\right)\cos 2\pi m_1x\\&=\sum_{m_1=-\lambda}^{\lambda}\left(\sum_{k={m_1}^2}^{\lambda^2}C(k-{m_1}^2)U(k)\right)\cos 2\pi m_1x,\end{aligned}$$

where $C(k)$ is the number of the points $(m_2,\cdots,m_n)\in\mathbb{Z}^{n-1}$ such that ${m_2}^2+\cdots+{m_n}^2=k$.

We can see the Pinsky phenomenon and the Gibbs phenomenon in our graphs. In the cases of $n=5$ and 6 we can see another phenomenon (see Figures 9, 10, 13 and 14).

Figure 1. $S_\lambda(F_{1/4} : x, 0, 0)$ for $-0.5 < x < 0.5$ in 3 dimensions.

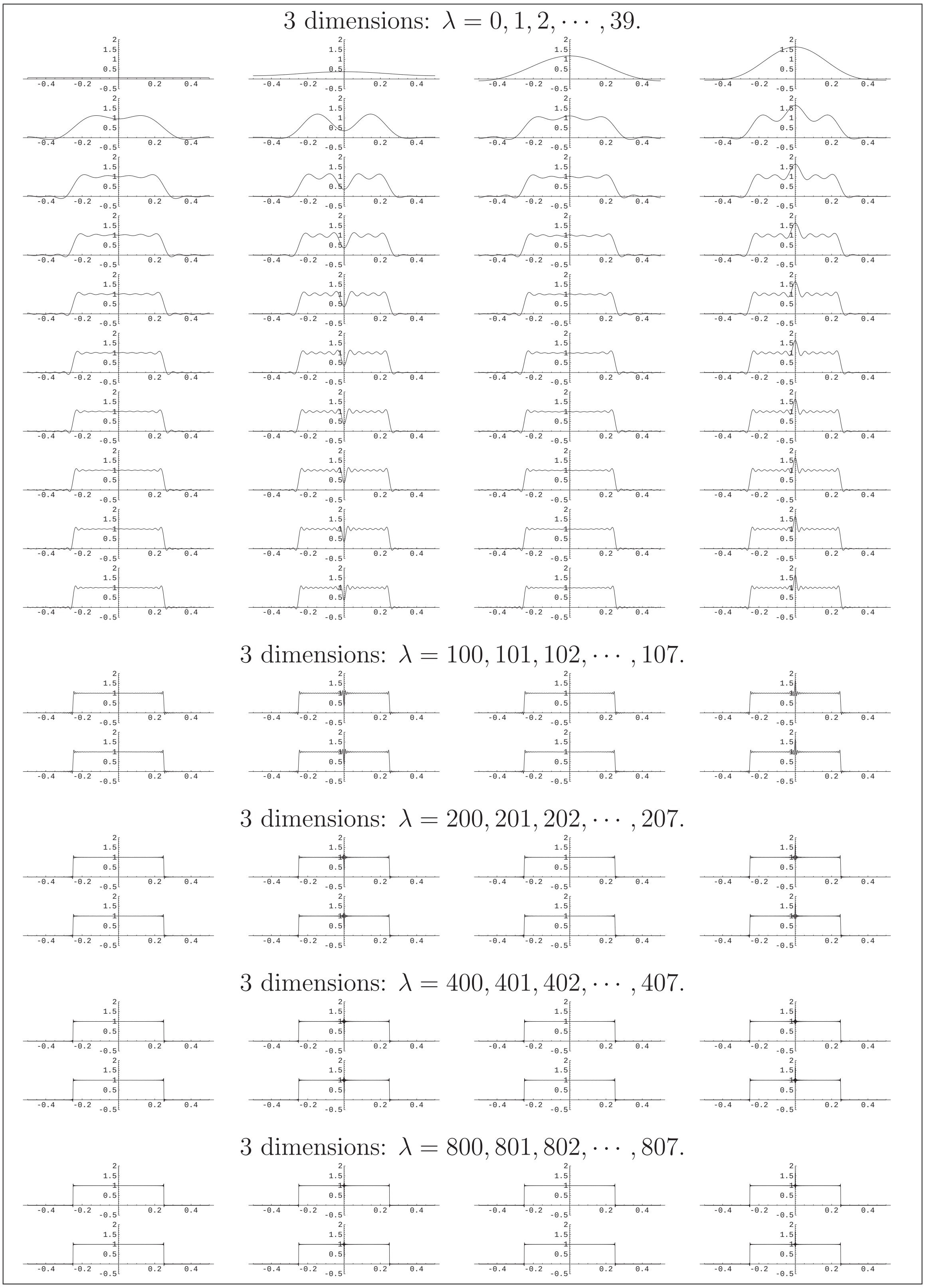


Figure 2. $S_\lambda(F_{1/4} : x, 0, 0, 0)$ for $-0.5 < x < 0.5$ in 4 dimensions.

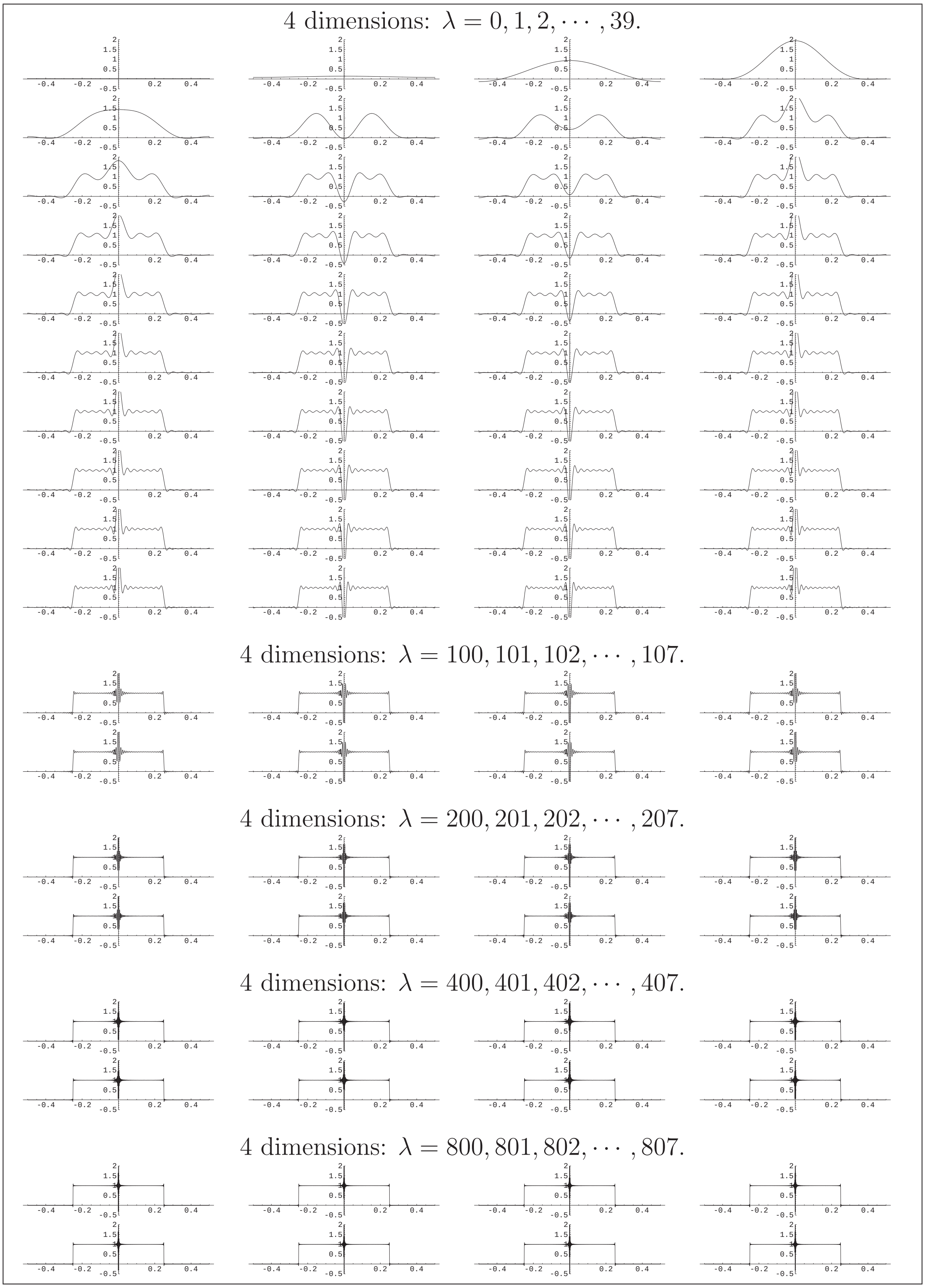


Figure 3. $S_\lambda(F_{1/4} : x, 0, 0, 0, 0)$ for $-0.5 < x < 0.5$ in 5 dimensions.

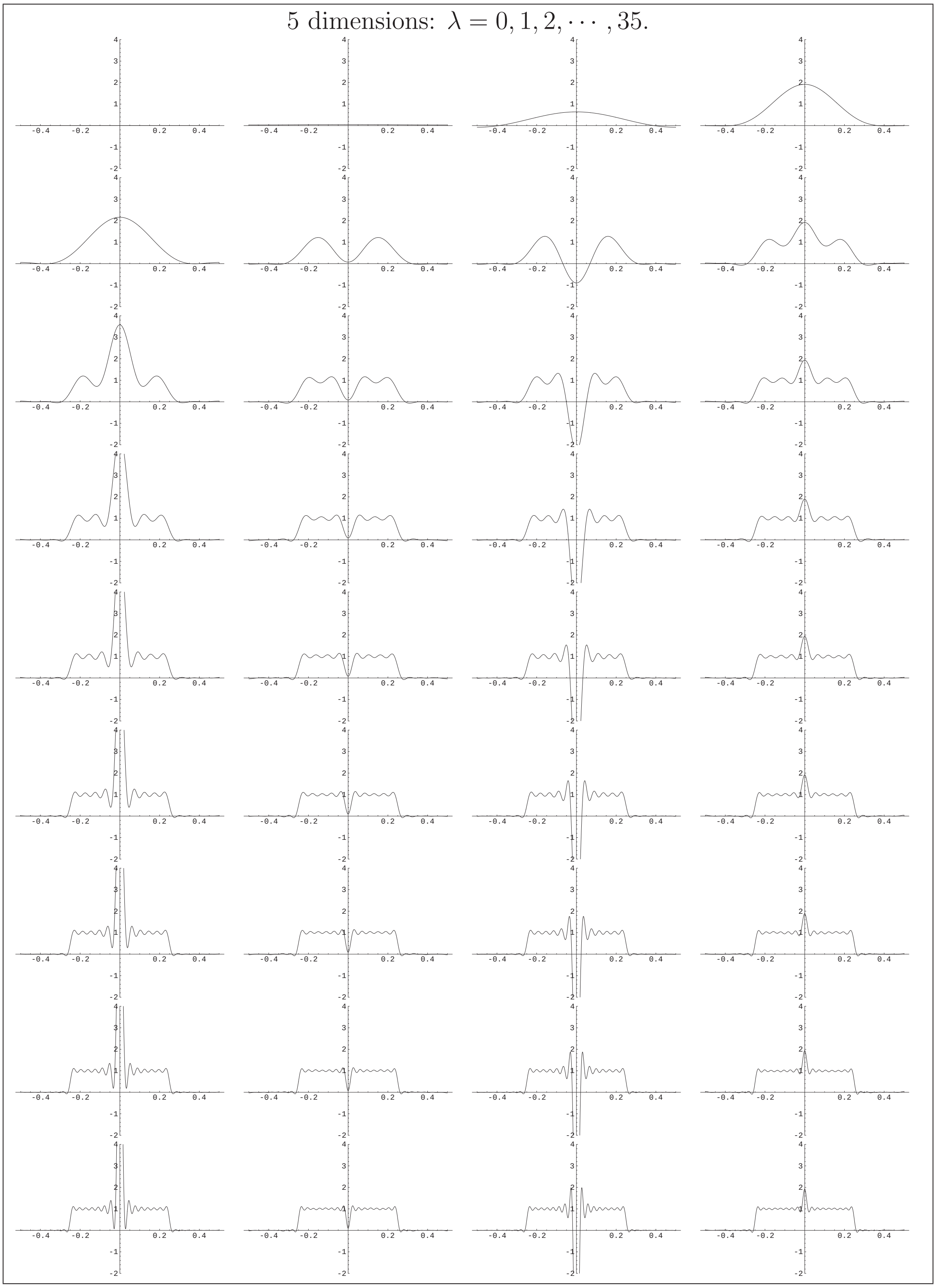


Figure 4. $S_\lambda(F_{1/4} : x, 0, 0, 0, 0)$ for $-0.5 < x < 0.5$ in 5 dimensions.

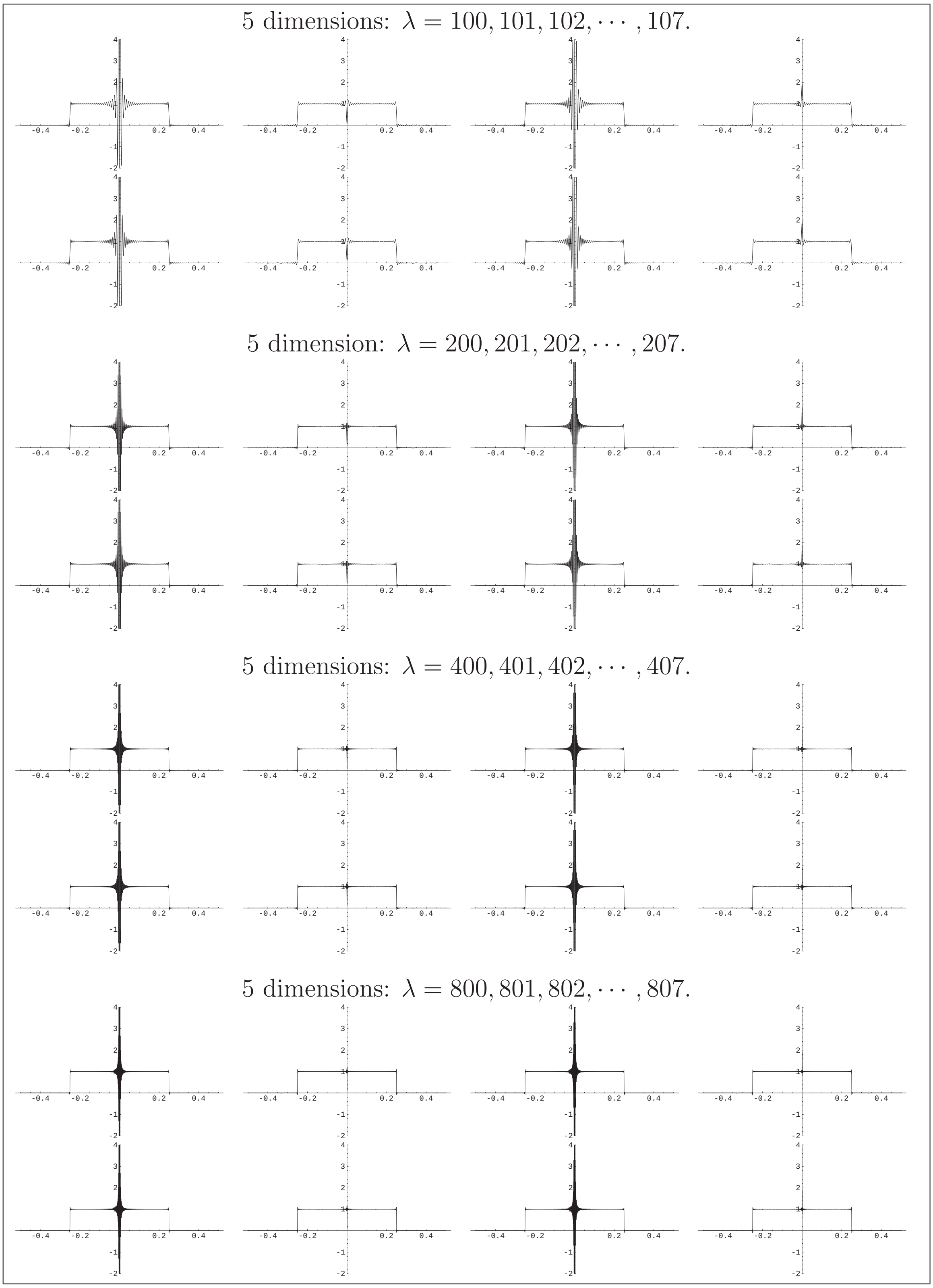


Figure 5. $S_\lambda(F_{1/4} : x, 0, 0, 0, 0, 0)$ for $-0.5 < x < 0.5$ in 6 dimensions.

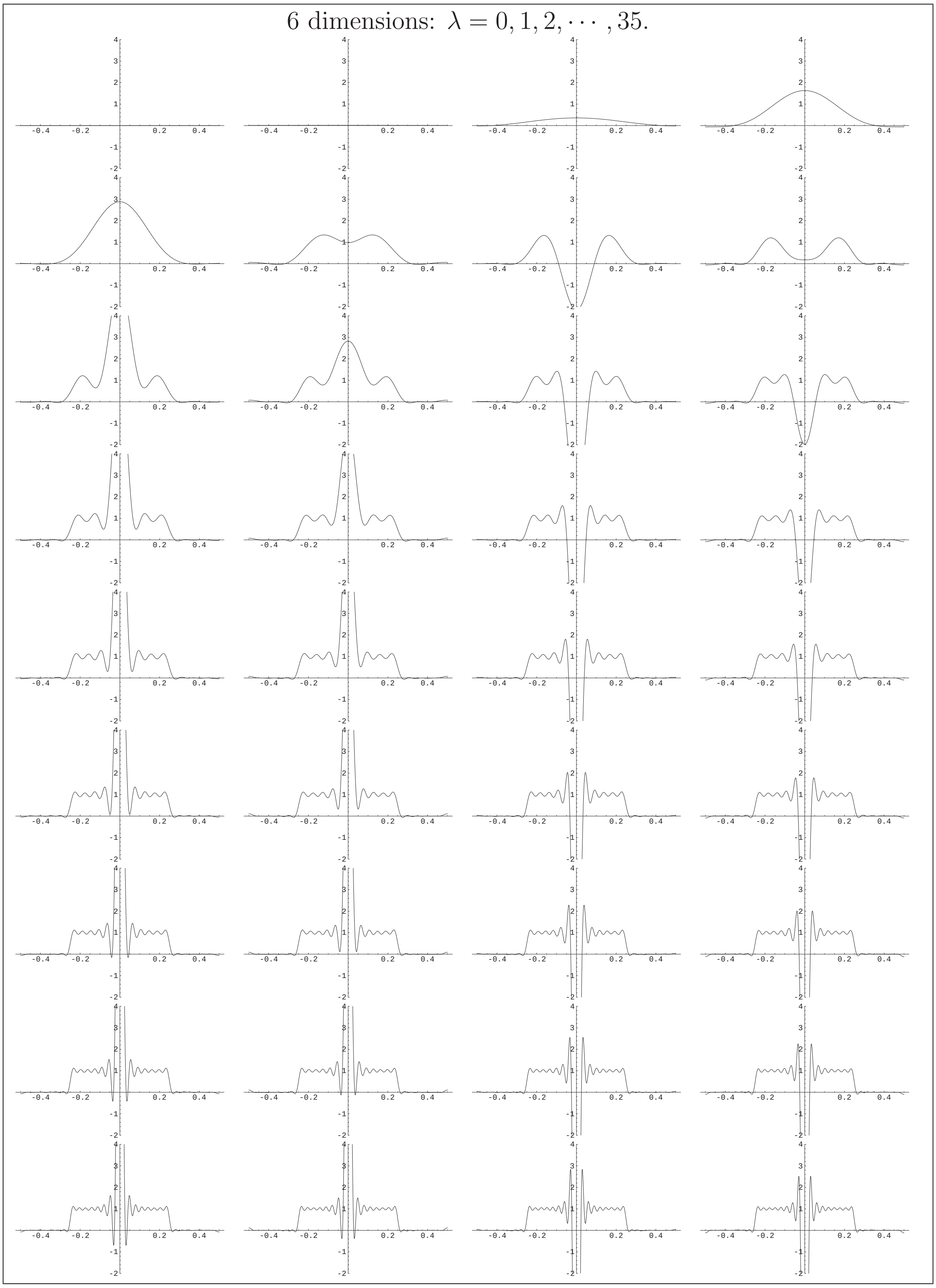


Figure 6. $S_\lambda(F_{1/4} : x, 0, 0, 0, 0, 0)$ for $-0.5 < x < 0.5$ in 6 dimensions.

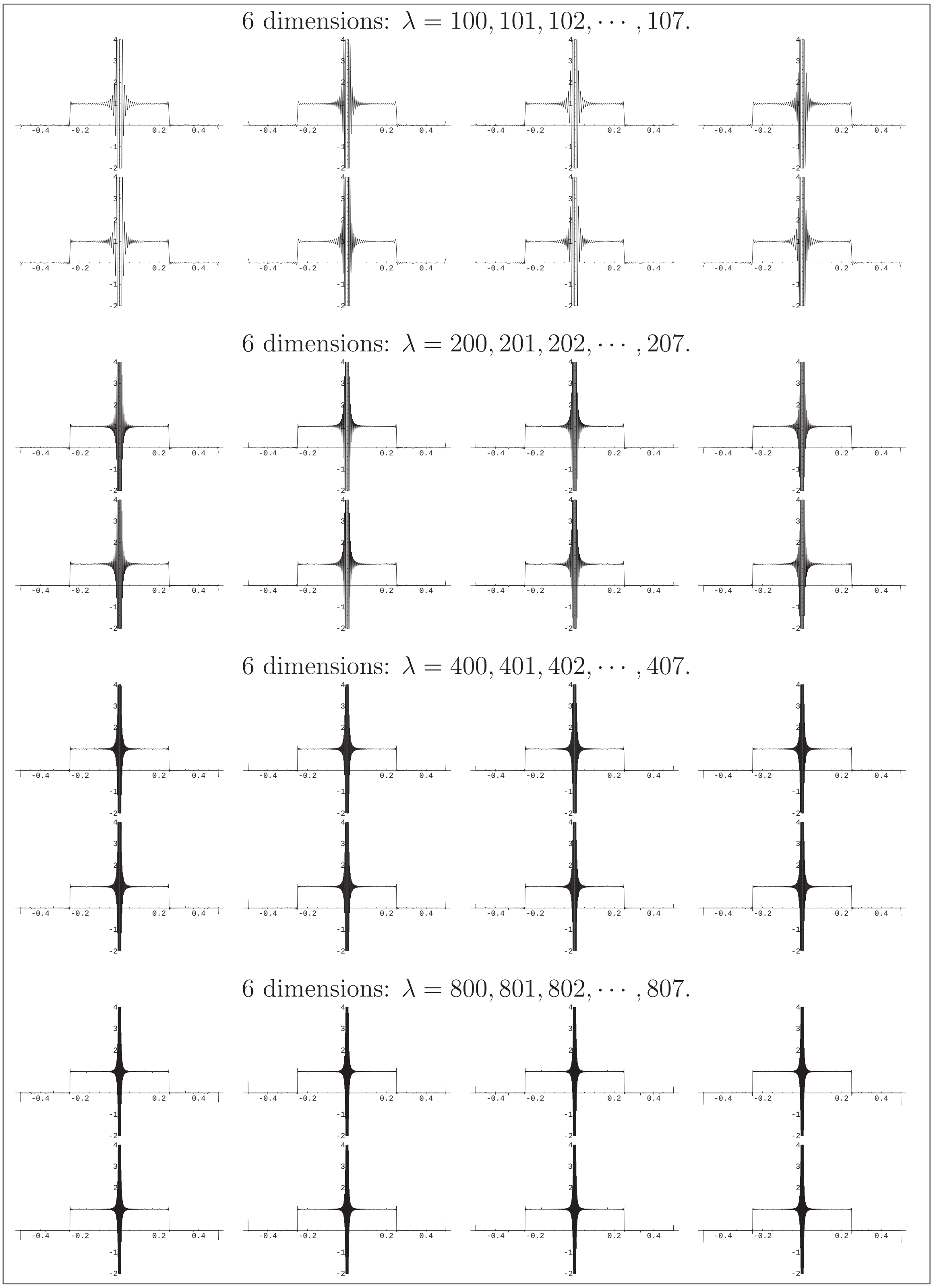


Figure 7.
$S_\lambda(F_{1/4} : x, 0, 0)$ for $0.2 < x < 0.5$.

3 dimensions: $\lambda = 400, 401, 402, \cdots, 407$.

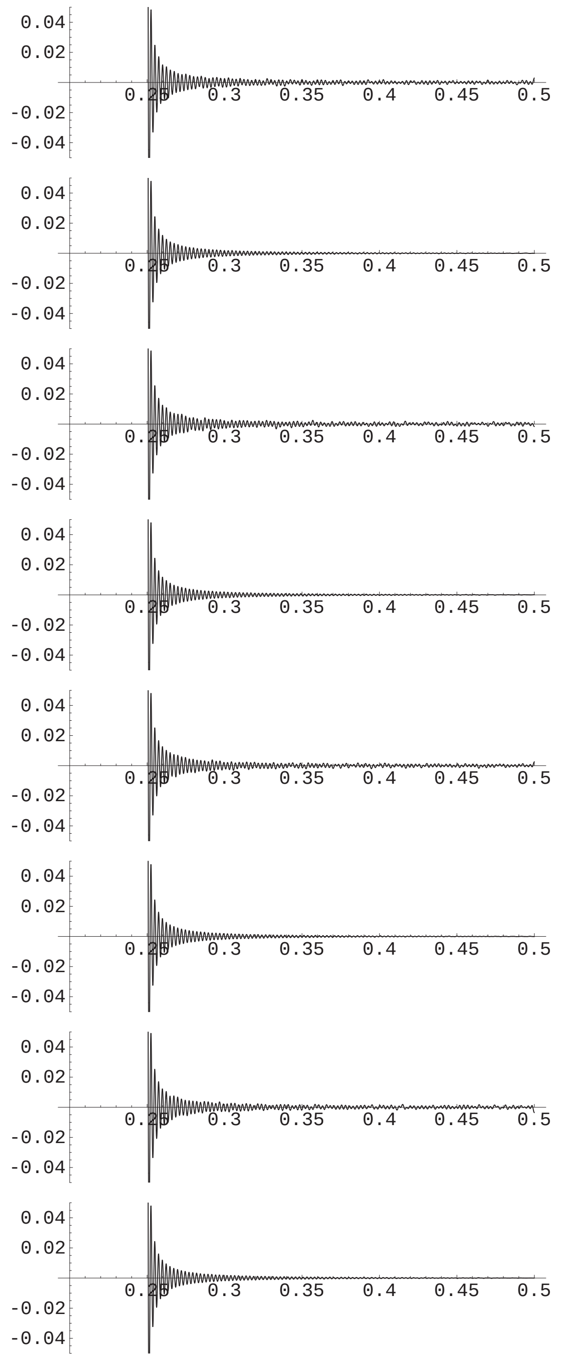


Figure 8.
$S_\lambda(F_{1/4} : x, 0, 0, 0)$ for $0.2 < x < 0.5$.

4 dimensions: $\lambda = 400, 401, 402, \cdots, 407$.

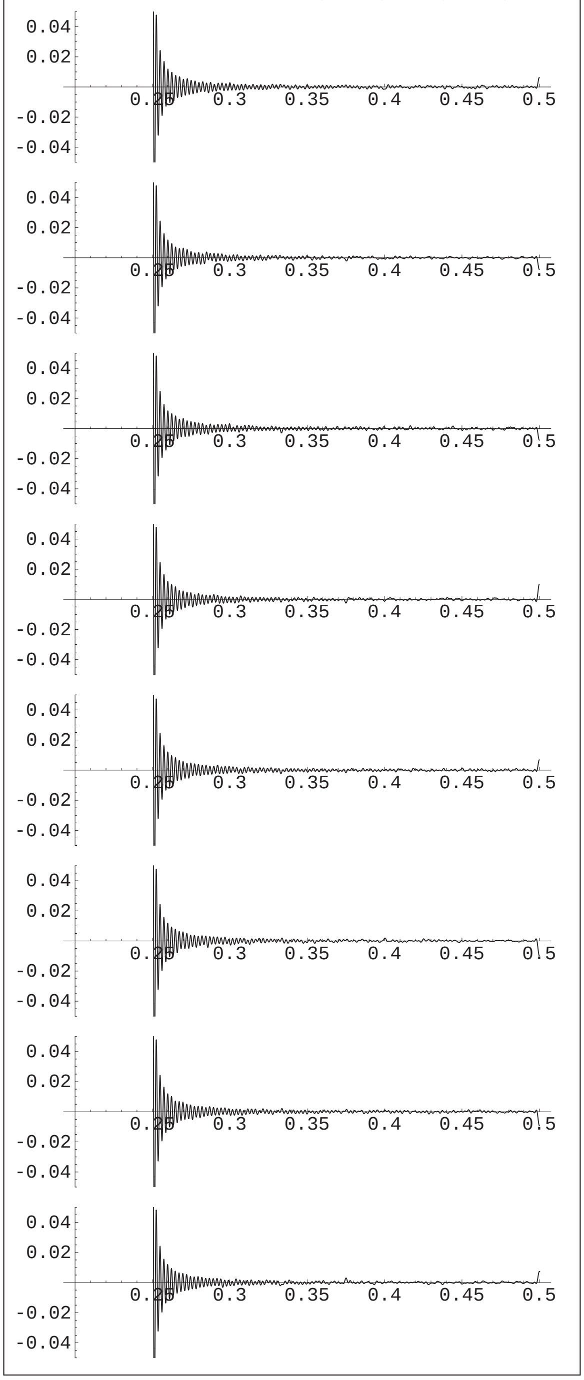


Figure 9.
$S_\lambda(F_{1/4} : x, 0, 0, 0, 0)$ for $0.2 < x < 0.5$.

5 dimensions: $\lambda = 400, 401, 402, \cdots, 407.$

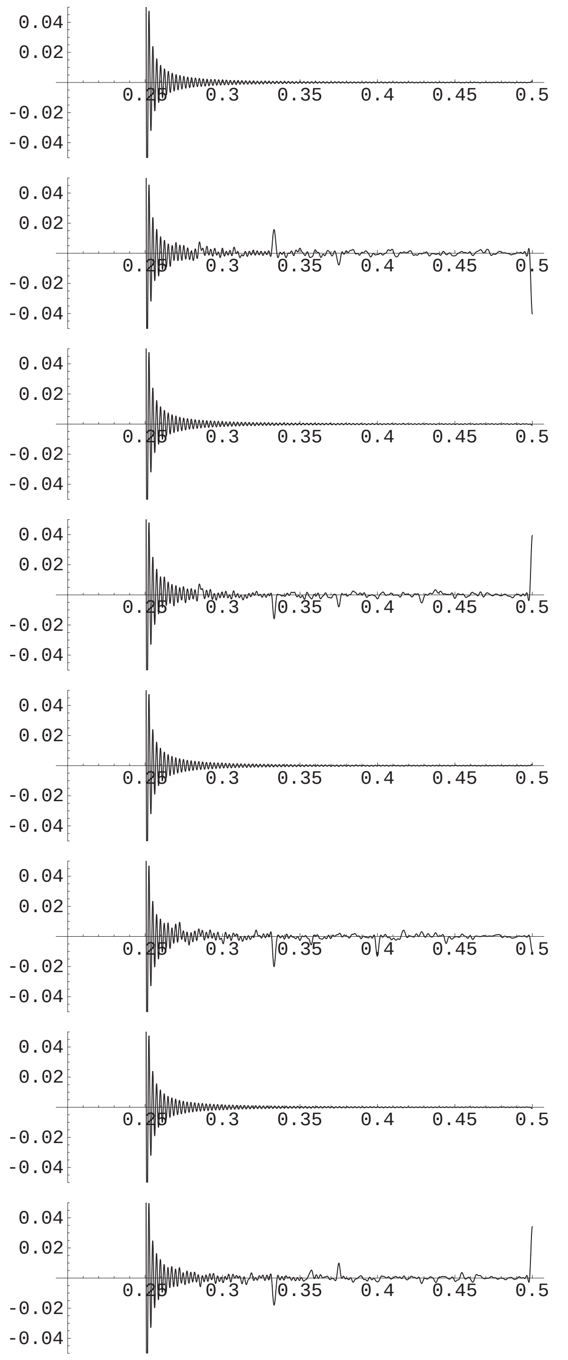


Figure 10.
$S_\lambda(F_{1/4} : x, 0, 0, 0, 0, 0)$ for $0.2 < x < 0.5$.

6 dimensions: $\lambda = 400, 401, 402, \cdots, 407.$

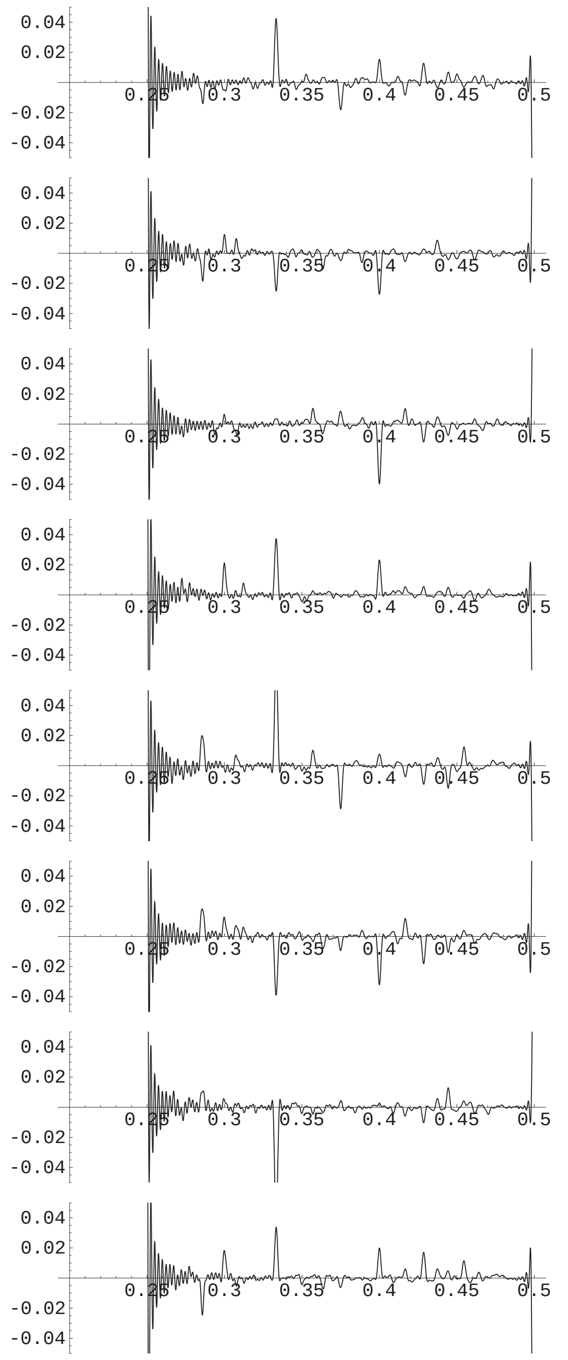


Figure 11.
$S_\lambda(F_{1/4} : x, 0, 0)$ for $0.2 < x < 0.5$.

3 dimensions: $\lambda = 800, 801, 802, \cdots, 807$.

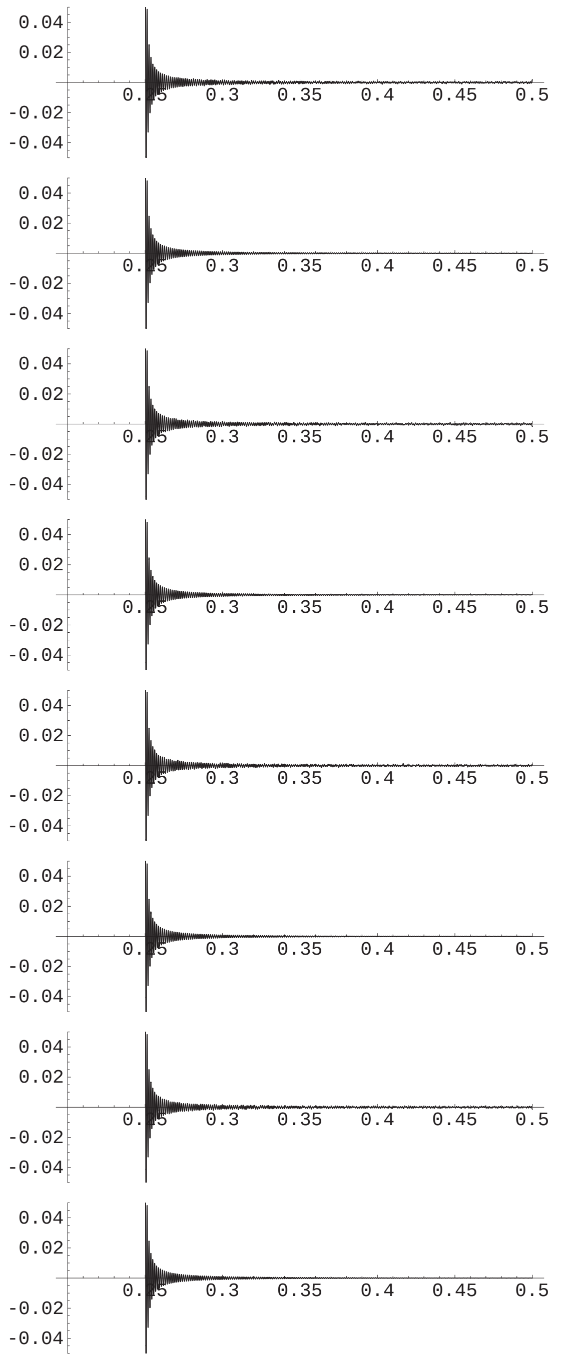


Figure 12.
$S_\lambda(F_{1/4} : x, 0, 0, 0)$ for $0.2 < x < 0.5$.

4 dimensions: $\lambda = 800, 801, 802, \cdots, 807$.

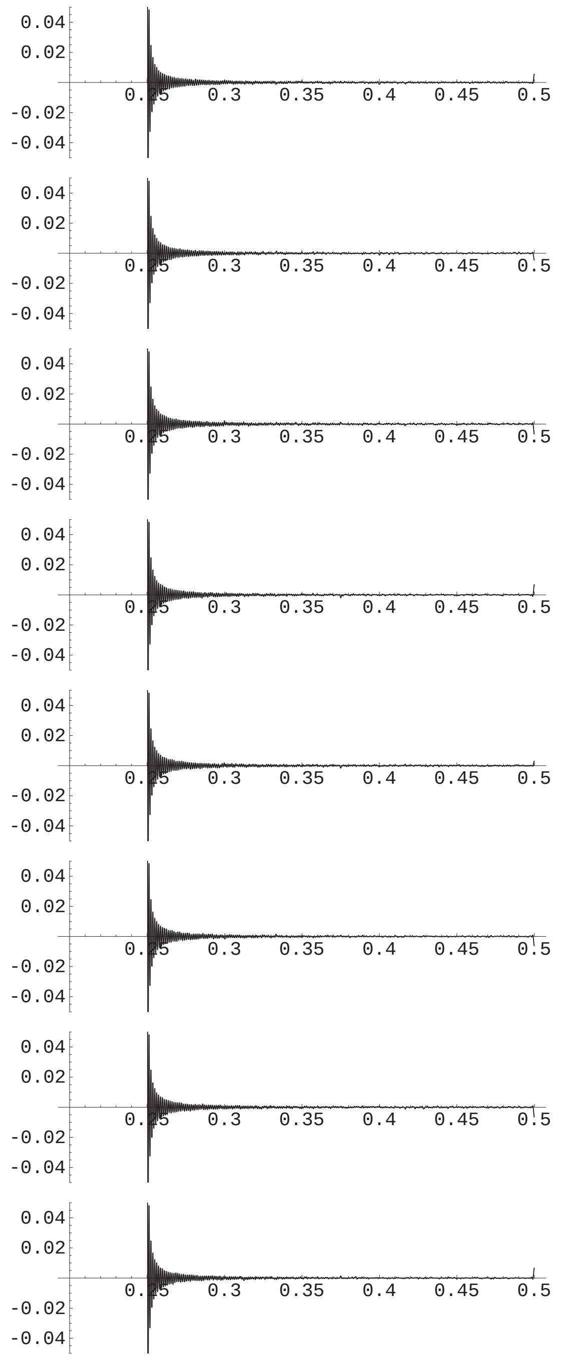


Figure 13.
$S_\lambda(F_{1/4} : x, 0, 0, 0, 0)$ for $0.2 < x < 0.5$.

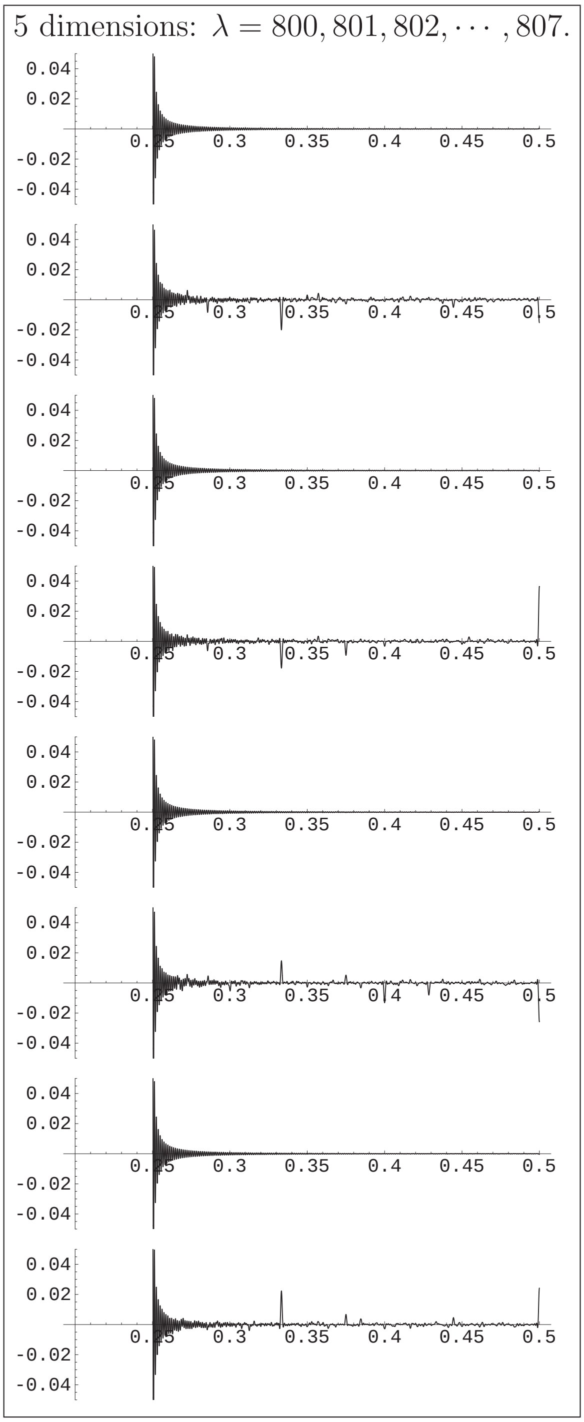
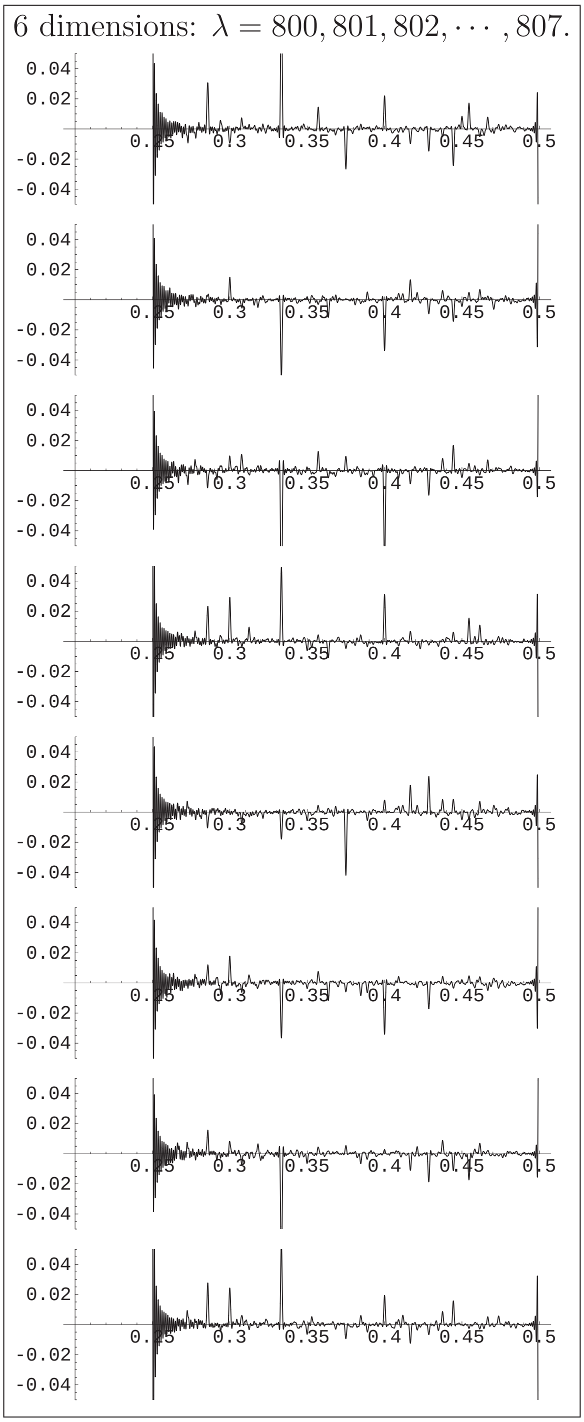


Figure 14.
$S_\lambda(F_{1/4} : x, 0, 0, 0, 0, 0)$ for $0.2 < x < 0.5$.



REFERENCES

[1] J. M. Ash, *Multiple Trigonometric Series*, Math. Assoc. Amer., Washington, D. C., **13** (1976), 76–96.

[2] J. M. Ash and G. Wang, *A Survey of Uniqueness Questions in Multiple Trigonometric Series*, Contemp. Math., **208** (1997), 35–71.

[3] C. P. Chen and P. H. Hsieh, *Pointwise Convergence of Double Trigonometric Series*, J. Math. Anal. Appl. **172** (1993), 582–599.

[4] C. Goffman and F. C. Liu, *On the localization property of square partial sums for multiple Fourier series*, Studia Math., **44** (1972), 61–69.

[5] G. H. Hardy and E. Landau, *The lattice points of a circle*, Proc. Roy. Soc. London Ser. A, **105** (1924), 244–258.

[6] C. Harada and E. Nakai, *The square partial sums of the Fourier transform of radial functions in three dimensions*, Sci. Math. Jpn., **55** (2002), 467–477. (Sci. Math. Jpn. Online, **5** (2001), 329–339.)

[7] M. Kojima, *On convergence of multiple Fourier series for piecewise smooth radial functions in* $[-\pi,\pi]^n$ $(n>3)$, Int. Info. Sci. Tohoku, **2** (1996), 103–110.

[8] S. Kuratsubo, *On pointwise convergence of Fourier series of indicator function of* n *dimensional ball*, Sci. Report Hirosaki, **43** (1996), 199–207.

[9] S. Kuratsubo, *On pointwise convergence of Fourier series of radial functions in several variables*, Proc. Amer. Math. Soc. **127** (1999), 2987–2994.

[10] M. A. Pinsky, *Pointwise Fourier inversion and related eigenfunction expansions*, Comm. Pure. Appl. Math.. **47** (1994), 653–681.

[11] M. A. Pinsky, *Fourier inversion in several variables – a new look at an old problem*, Representation theory and harmonic analysis (Cincinnati, OH, 1994), 139–152, Contemp. Math., 191, Amer. Math. Soc., Providence, RI, 1995.

[12] M. A. Pinsky, Introduction to Fourier analysis and wavelets. Brooks/Cole Series in Advanced Mathematics. Brooks/Cole, Pacific Grove, CA, 2002.

[13] M. A. Pinsky, N. K. Stanton and P. E. Trapa, *Fourier series of radial functions in several variables*, J. Funct. Anal. **116** (1993), 111–132.

[14] A. Zygmund, *Notes on the History of Fourier Series*, Math. Assoc. Amer., Washington, D. C., **13** (1976), 1–19.

特性関数の多変数フーリエ級数におけるピンスキー現象について

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$n \geq 3$ とする。n 次元球の特性関数について、フーリエ級数の球形部分和は球の中心で振動すること、そしてその振動の周期は、球の半径が $1/k$ $(k=2,3,\cdots)$ のとき k であることを指摘する。さらに、$n=3,4,5,6$ の場合について、その部分和のグラフを与える。グラフには、ピンスキー現象とギッブス現象が見られる。$n=5$ と 6 の場合には、これ以外の現象も見ることができる。

キーワード: フーリエ級数, ピンスキー現象, ギッブス現象, 球形部分和, 放射状関数, 特性関数