

# Pointwise Multipliers on the Lorentz Spaces

Eiichi NAKAI

*Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582, Japan*

(Received April 26, 1996)

**Abstract :**  $L^p$ -spaces ( $0 < p \leq \infty$ ) are complete quasi-normed linear spaces. A function  $g$  is called a pointwise multiplier from  $L^{p_1}$  to  $L^{p_2}$ , if the pointwise multiplication  $fg$  belongs to  $L^{p_2}$  for each  $f \in L^{p_1}$ . We denote by  $PWM(L^{p_1}, L^{p_2})$  the set of all pointwise multipliers from  $L^{p_1}$  to  $L^{p_2}$ . It is known that, if  $1/p_1 + 1/p_3 = 1/p_2$ , then

$$PWM(L^{p_1}, L^{p_2}) = L^{p_3} \quad \text{and} \quad \|g\|_{\text{Op}} = \|g\|_{L^{p_3}},$$

where  $\|g\|_{\text{Op}}$  is the operator norm of  $g \in PWM(L^{p_1}, L^{p_2})$ . The purpose of this paper is to generalize the above equalities to the Lorentz spaces.

**Key Words :** multiplier, pointwise multiplier, Lorentz space

## I. Introduction

Let  $E$  and  $F$  be spaces of real- or complex-valued functions defined on a set  $X$ . A real- or complex-valued function  $g$  defined on  $X$  is called a pointwise multiplier from  $E$  to  $F$ , if the pointwise multiplication  $fg$  belongs to  $F$  for each  $f \in E$ . We denote by  $PWM(E, F)$  the set of all pointwise multipliers from  $E$  to  $F$ .

$L^p$ -spaces ( $0 < p \leq \infty$ ) on a measure space  $X$  are complete quasi-normed linear spaces. If  $X$  is  $\sigma$ -finite, then it is known that

$$(1) \quad PWM(L^{p_1}(X), L^{p_2}(X)) = L^{p_3}(X), \quad 1/p_1 + 1/p_3 = 1/p_2,$$

and

$$(2) \quad \|g\|_{\text{Op}} = \|g\|_{L^{p_3}},$$

where  $\|g\|_{\text{Op}}$  is the operator norm of  $g \in PWM(L^{p_1}(X), L^{p_2}(X))$ , i.e.

$$\|g\|_{\text{Op}} = \inf\{\beta > 0 : \|fg\|_{L^{p_2}} \leq \beta \|f\|_{L^{p_1}} \text{ for all } f \in L^{p_1}(X)\}.$$

The purpose of this paper is to generalize the equalities (1) and (2) to the Lorentz spaces.

Let  $E = L^p(X)$ . Then  $E$  has the following properties.

$$(3) \quad f \in E \text{ and } |h(x)| \leq |f(x)| \text{ a.e. } X \Rightarrow h \in E \text{ and } \|h\| \leq \|f\|,$$

$$(4) \quad f_i \in E, \quad \|f_i\| \leq M < \infty \quad (i = 1, 2, \dots), \quad f_1(x) \leq f_2(x) \leq \dots, \\ \text{and } f_i(x) \rightarrow f(x) \quad (i \rightarrow \infty) \text{ a.e. } X \quad \Rightarrow \quad f \in E.$$

Let  $(X, \mu)$  be a  $\sigma$ -finite measure space, i.e.  $X$  is expressible as a countable union of sets  $X_i \subset X$  such that  $\mu(X_i) < \infty$  ( $i = 1, 2, \dots$ ). Then

$$(5) \quad \text{any bounded function whose support is included in } X_i \text{ for some } i \text{ is in } E.$$

We use the following theorems.

**Theorem A** ([5]). *Let  $(X, \mu)$  be a  $\sigma$ -finite measure space and let  $X$  be a union of sets  $X_i \subset X$  such that  $\mu(X_i) < \infty$  ( $i = 1, 2, \dots$ ). Let  $E_i$  ( $i = 1, 2, 3$ ) be sets of measurable functions. Suppose that  $E_1$  is a complete quasi-normed linear space, that  $E_2$  is a quasi-normed linear space with the property (3), that  $E_3$  is a quasi-normed linear space with the properties (4) and (5), and that*

$$PWM(E_1, E_2) \supset E_3.$$

*If every  $g \in E_3 \subset PWM(E_1, E_2)$  is a bounded operator and the operator norm is comparable to  $\|g\|_{E_3}$ , then*

$$PWM(E_1, E_2) = E_3.$$

**Theorem B** ([4]). *Let  $(X, \mu)$  be a  $\sigma$ -finite measure space and let  $X$  be a union of sets  $X_i \subset X$  such that  $\mu(X_i) < \infty$  ( $i = 1, 2, \dots$ ). Let  $E$  be a set of measurable functions and a complete quasi-normed linear space with the property (3). If the characteristic function of  $X_i$  is in  $E$  for  $i = 1, 2, \dots$ , then*

$$PWM(E, E) = L^\infty(X) \quad \text{and} \quad \|g\|_{OP} = \|g\|_{L^\infty}.$$

## II. Main results

Let  $X = (X, \mu)$  be a measure space. First, we recall the definitions of the Lorentz spaces. For a measurable function  $f$ , the distribution function  $\mu(f, s)$  and the rearrangement  $f^*(t)$  are defined by

$$\mu(f, s) = \mu(\{x \in X : |f(x)| > s\}), \quad \text{for } s > 0, \\ f^*(t) = \inf\{s > 0 : \mu(f, s) \leq t\}, \quad \text{for } t > 0.$$

The Lorentz space  $L^{(p, q)}(X)$  is defined to be the set of all  $f$  such that  $\|f\|_{(p, q)} < \infty$ , where

$$\|f\|_{(p, q)} = \begin{cases} \left( \int_0^\infty t^{(q/p)-1} (f^*(t))^q dt \right)^{1/q}, & 0 < p \leq \infty, 0 < q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t), & 0 < p \leq \infty, q = \infty. \end{cases}$$

If  $p = \infty$  and  $0 < q < \infty$ , then  $L^{(p, q)}(X) = \{0\}$ . Note that

$$L^{(p, p)}(X) = L^p(X) \quad \text{and} \quad \|f\|_{(p, p)} = \|f\|_{L^p}, \quad 0 < p \leq \infty.$$

In general,  $(L^{(p,q)}(X), \|\cdot\|_{(p,q)})$  are complete quasi-normed linear spaces.

Let

$$f^{(*,r)}(t) = \begin{cases} \left( \frac{1}{t} \int_0^t (f^*(s))^r ds \right)^{1/r}, & 0 < r < \infty, \\ \|f\|_{L^\infty}, & r = \infty. \end{cases}$$

Now we define that  $L^{(p,q,r)}(X)$  is the set of all  $f$  such that  $\|f\|_{(p,q,r)} < \infty$ , where

$$\|f\|_{(p,q,r)} = \begin{cases} \left( \int_0^\infty t^{(q/p)-1} (f^{(*,r)}(t))^q dt \right)^{1/q}, & 0 < p, r \leq \infty, 0 < q < \infty, \\ \sup_{t>0} t^{1/p} f^{(*,r)}(t), & 0 < p, r \leq \infty, q = \infty. \end{cases}$$

If  $p \leq r$  and  $0 < q < \infty$ , or if  $p < r$  and  $q = \infty$ , then  $L^{(p,q,r)}(X) = \{0\}$ . So we may assume that

$$(6) \quad 0 < r < p < \infty, 0 < q < \infty,$$

$$(7) \quad 0 < r \leq p < \infty, q = \infty,$$

$$(8) \quad 0 < r \leq p = q = \infty.$$

In the case (8),  $\|f\|_{(p,q,r)} = \|f\|_{L^\infty}$ .

Since  $f^*(t)$  is non-increasing,  $f^*(t) \leq f^{(*,r)}(t)$ . Hence  $L^{(p,q,r)}(X) \subset L^{(p,q)}(X)$ . If  $0 < r < p < \infty$  and  $r \leq q \leq \infty$ , then  $L^{(p,q,r)}(X) = L^{(p,q)}(X)$  and

$$(9) \quad \|f\|_{L^{(p,q)}} \leq \|f\|_{(p,q,r)} \leq (p/(p-r))^{1/r} \|f\|_{L^{(p,q)}}.$$

Actually, if  $q = \infty$ , then

$$\begin{aligned} t^{1/p} \left( \frac{1}{t} \int_0^t (f^*(s))^r ds \right)^{1/r} &= t^{1/p-1/r} \left( \int_0^t s^{-r/p} (s^{1/p} f^*(s))^r ds \right)^{1/r} \\ &\leq t^{1/p-1/r} \left( \int_0^t s^{-r/p} ds \right)^{1/r} \sup_{t>0} t^{1/p} f^*(t) \leq \left( \frac{p}{p-r} \right)^{1/r} \sup_{t>0} t^{1/p} f^*(t). \end{aligned}$$

For the case  $q < \infty$ , see [2].

$(L^{(p,q,r)}(X), \|\cdot\|_{(p,q,r)})$  are complete quasi-normed linear spaces. If  $1 < p < \infty$ ,  $1 \leq q \leq \infty$  and  $r = 1$ , then they are Banach spaces ([6]).

The spaces  $L^{(p,q,r)}(X)$  have the properties (3) and (4). If  $X$  is  $\sigma$ -finite, then they have the property (5).

Our main results are the following.

**Theorem 1.** *Let  $(X, \mu)$  be a  $\sigma$ -finite measure space. If the pairs*

*$(p_i, q_i, r_i) (i = 1, 2, 3)$  satisfy (6), (7) or (8), and if*

$$(10) \quad 1/p_1 + 1/p_3 = 1/p_2 \text{ and } p_1 : p_2 : p_3 = q_1 : q_2 : q_3 = r_1 : r_2 : r_3,$$

*then*

$$PWM(L^{(p_1, q_1, r_1)}(X), L^{(p_2, q_2, r_2)}(X)) = L^{(p_3, q_3, r_3)}(X),$$

*and*

$$\|g\|_{Op} = \|g\|_{(p_3, q_3, r_3)},$$

where  $\|g\|_{\text{Op}}$  is the operator norm of  $g \in PWM(L^{(p_1, q_1, r_1)}(X), L^{(p_2, q_2, r_2)}(X))$ .

*Remark.* The assumption (10) includes the following cases.

$$\begin{aligned} p_1 = q_1 = r_1 = \infty \quad \text{and} \quad 0 < p_2 = p_3, \quad q_2 = q_3, \quad r_2 = r_3 < \infty, \\ p_3 = q_3 = r_3 = \infty \quad \text{and} \quad 0 < p_1 = p_2, \quad q_1 = q_2, \quad r_1 = r_2 < \infty, \\ 0 < p_i, r_i < q_i = \infty \quad (i = 1, 2, 3), \quad 1/p_1 + 1/p_3 = 1/p_2 \\ \text{and} \quad p_1 : p_2 : p_3 = r_1 : r_2 : r_3, \\ p_i = q_i = r_i = \infty \quad (i = 1, 2, 3). \end{aligned}$$

**Corollary 2.** Let  $(X, \mu)$  be a  $\sigma$ -finite measure space. If the pairs  $(p_i, q_i, r_i)$  ( $i = 1, 2, 3$ ) satisfy (10) and  $0 < r_i < p_i < \infty$ ,  $r_i \leq q_i \leq \infty$  ( $i = 1, 2, 3$ ), then

$$\begin{aligned} PWM((L^{(p_1, q_1)}(X), \|\cdot\|_{(p_1, q_1, r_1)}), (L^{(p_2, q_2)}(X), \|\cdot\|_{(p_2, q_2, r_2)})) \\ = (L^{(p_3, q_3)}(X), \|\cdot\|_{(p_3, q_3, r_3)}), \end{aligned}$$

and

$$\|g\|_{\text{Op}} = \|g\|_{(p_3, q_3, r_3)},$$

where  $\|g\|_{\text{Op}}$  is the operator norm of

$$g \in PWM((L^{(p_1, q_1)}(X), \|\cdot\|_{(p_1, q_1, r_1)}), (L^{(p_2, q_2)}(X), \|\cdot\|_{(p_2, q_2, r_2)})).$$

**Corollary 3.** Let  $(X, \mu)$  be a  $\sigma$ -finite measure space. Let  $0 < p_i < \infty$  and  $0 < q_i \leq \infty$ . If  $1/p_1 + 1/p_3 = 1/p_2$  and  $p_1 : p_2 : p_3 = q_1 : q_2 : q_3$ , then

$$\begin{aligned} PWM((L^{(p_1, q_1)}(X), \|\cdot\|_{(p_1, q_1)}), (L^{(p_2, q_2)}(X), \|\cdot\|_{(p_2, q_2)})) \\ = (L^{(p_3, q_3)}(X), \|\cdot\|_{(p_3, q_3)}), \end{aligned}$$

and

$$\|g\|_{(p_3, q_3)} \leq \|g\|_{\text{Op}} \leq e^{1/p_2} \|g\|_{(p_3, q_3)},$$

where  $\|g\|_{\text{Op}}$  is the operator norm of

$$g \in PWM((L^{(p_1, q_1)}(X), \|\cdot\|_{(p_1, q_1)}), (L^{(p_2, q_2)}(X), \|\cdot\|_{(p_2, q_2)})).$$

[5] has shown that, if

$$g \in PWM((L^{(p_1, q_1)}(X), \|\cdot\|_{(p_1, q_1)}), (L^{(p_2, q_2)}(X), \|\cdot\|_{(p_2, q_2)})),$$

then  $\|g\|_{\text{Op}} = \|g\|_{(p_3, q_3)}$  for  $p_2 \geq q_2$ . However, for  $p_2 < q_2$ ,

$$\|g\|_{(p_3, q_3)} < \|g\|_{\text{Op}} \leq e^{1/p_2} \|g\|_{(p_3, q_3)}.$$

### III. Proofs

To prove Theorem 1, we show the following lemma.

**Lemma 4.** Let  $1/p_1 + 1/p_3 = 1/p_2$ ,  $1/q_1 + 1/q_3 = 1/q_2$  and  $1/r_1 + 1/r_3 = 1/r_2$ . If  $f \in L^{(p_1, q_1, r_1)}(X)$  and  $g \in L^{(p_3, q_3, r_3)}(X)$ , then  $fg \in L^{(p_2, q_2, r_2)}(X)$  and

$$(11) \quad \|fg\|_{(p_2, q_2, r_2)} \leq \|f\|_{(p_1, q_1, r_1)} \|g\|_{(p_3, q_3, r_3)}.$$

*Proof.* First we note that

$$\int_0^t (fg)^*(s) ds \leq \int_0^t f^*(s) g^*(s) ds$$

(see for example [2] (1.9), p.257). Since  $(f^*(s))^r = (|f|^r)^*(s)$ ,

$$\int_0^t ((fg)^*(s))^r ds \leq \int_0^t (f^*(s) g^*(s))^r ds.$$

Case 1:  $0 < p_i, q_i, r_i < \infty (i = 1, 2, 3)$ . By Hölder's inequality, we have

$$\begin{aligned} & \left( \int_0^\infty t^{(q_2/p_2)-1} \left( \frac{1}{t} \int_0^t ((fg)^*(s))^{r_2} ds \right)^{q_2/r_2} dt \right)^{1/q_2} \\ & \leq \left( \int_0^\infty t^{(q_2/p_2)-1} \left( \frac{1}{t} \int_0^t (f^*(s) g^*(s))^{r_2} ds \right)^{q_2/r_2} dt \right)^{1/q_2} \\ & \leq \left( \int_0^\infty t^{(q_2/p_2)-1} \left( \frac{1}{t} \int_0^t (f^*(s))^{r_1} ds \right)^{q_2/r_1} \left( \frac{1}{t} \int_0^t (g^*(s))^{r_3} ds \right)^{q_2/r_3} dt \right)^{1/q_2} \\ & \leq \left( \int_0^\infty t^{(q_1/p_1)-1} \left( \frac{1}{t} \int_0^t (f^*(s))^{r_1} ds \right)^{q_1/r_1} dt \right)^{1/q_1} \\ & \quad \times \left( \int_0^\infty t^{(q_3/p_3)-1} \left( \frac{1}{t} \int_0^t (g^*(s))^{r_3} ds \right)^{q_3/r_3} dt \right)^{1/q_3}. \end{aligned}$$

Hence we have (11).

Case 2:  $q_1 = \infty$  and  $0 < q_2 = q_3 < \infty$ .

$$\begin{aligned} & \left( \int_0^\infty t^{(q_2/p_2)-1} \left( \frac{1}{t} \int_0^t (f^*(s))^{r_1} ds \right)^{q_2/r_1} \left( \frac{1}{t} \int_0^t (g^*(s))^{r_3} ds \right)^{q_2/r_3} dt \right)^{1/q_2} \\ & \leq \left( \sup_{t>0} t^{1/p_1} \left( \frac{1}{t} \int_0^t (f^*(s))^{r_1} ds \right)^{1/r_1} \right) \\ & \quad \times \left( \int_0^\infty t^{(q_3/p_3)-1} \left( \frac{1}{t} \int_0^t (g^*(s))^{r_3} ds \right)^{q_3/r_3} dt \right)^{1/q_3}. \end{aligned}$$

Hence we have (11).

Case 3:  $0 < q_1 = q_2 < \infty$  and  $q_3 = \infty$ . In the same way as Case 2, we have (11).

Case 4:  $q_1 = q_2 = q_3 = \infty$ .

$$\begin{aligned} & \sup_{t>0} \left( t^{1/p_2} \left( \frac{1}{t} \int_0^t (f^*(s))^{r_1} ds \right)^{1/r_1} \left( \frac{1}{t} \int_0^t (g^*(s))^{r_3} ds \right)^{1/r_3} \right) \\ & \leq \left( \sup_{t>0} t^{1/p_1} \left( \frac{1}{t} \int_0^t (f^*(s))^{r_1} ds \right)^{1/r_1} \right) \left( \sup_{t>0} t^{1/p_3} \left( \frac{1}{t} \int_0^t (g^*(s))^{r_3} ds \right)^{1/r_3} \right). \end{aligned}$$

Hence we have (11).

Case 5:  $0 < r_i \leq p_i = q_i = \infty (i = 1, 2, 3)$ . Since  $\|fg\|_{L^\infty} \leq \|f\|_{L^\infty} \|g\|_{L^\infty}$ , we have (11).  $\square$

*Proof of Theorem 1.* By Lemma 4, we have

$$\begin{aligned} PWM(L^{(p_1, q_1, r_1)}(X), L^{(p_2, q_2, r_2)}(X)) & \supset L^{(p_3, q_3, r_3)}(X), \\ \|g\|_{Op} & \leq \|g\|_{(p_3, q_3, r_3)}. \end{aligned}$$

Let  $g \in L^{(p_3, q_3, r_3)}(X)$ . For  $0 < p_3 < \infty$ , let  $f = |g|^{p_3/p_1}$  ( $f \equiv 1$  for  $p_1 = \infty$ ). Then  $f$  is in  $L^{(p_1, q_1, r_1)}(X)$  and

$$\|fg\|_{(p_2, q_2, r_2)} = \|f\|_{(p_1, q_1, r_1)} \|g\|_{(p_3, q_3, r_3)}.$$

Hence

$$\|g\|_{\text{Op}} = \|g\|_{(p_3, q_3, r_3)}.$$

Therefore, by Theorem A, we have Theorem 1 for  $0 < p_3 < \infty$ . If  $p_3 = \infty$ , then  $p_1 = p_2$ ,  $q_1 = q_2$  and  $r_1 = r_2$ . This case is in Theorem B.  $\square$

*Proof of Corollary 2.* Since  $L^{(p, q, r)}(X) = L^{(p, q)}(X)$  for  $0 < r < p < \infty$  and  $r \leq q \leq \infty$ , Corollary 2 is immediate consequence to Theorem 1.  $\square$

*Proof of Corollary 3.* Choose  $r_i (i = 1, 2, 3)$  such that  $0 < r_i < p_i < \infty$ ,  $r_i \leq q_i \leq \infty$  and  $p_1 : p_2 : p_3 = r_1 : r_2 : r_3$ . By Lemma 4 and (9), we have

$$\begin{aligned} \|fg\|_{L^{(p_2, q_2)}} &\leq \|fg\|_{(p_2, q_2, r_2)} \leq \|f\|_{(p_1, q_1, r_1)} \|g\|_{(p_3, q_3, r_3)} \\ &\leq (p_1/(p_1 - r_1))^{1/r_1} (p_3/(p_3 - r_3))^{1/r_3} \|f\|_{L^{(p_1, q_1)}} \|g\|_{L^{(p_3, q_3)}}. \end{aligned}$$

Let  $r_1 \rightarrow 0$  and  $r_3 \rightarrow 0$ . Then

$$(p_1/(p_1 - r_1))^{1/r_1} (p_3/(p_3 - r_3))^{1/r_3} \rightarrow e^{1/p_1} e^{1/p_3} = e^{1/p_2}.$$

Hence

$$\|fg\|_{L^{(p_2, q_2)}} \leq e^{1/p_2} \|f\|_{L^{(p_1, q_1)}} \|g\|_{L^{(p_3, q_3)}}.$$

Therefore, in the same way as Theorem 1, we have Corollary 3.  $\square$

## References

- [1] J. Bergh and J. Löfström, *Interpolation spaces*, Springer-Verlag, Berlin Heidelberg New York, 1976.
- [2] R. A. Hunt, *On  $L(p, q)$  spaces*, Enseignement Math. 12(1966), 249–276.
- [3] G. G. Lorentz, *Some new functional spaces*, Ann. of Math. 51 (1950), 37–55.
- [4] E. Nakai, *Pointwise multipliers on some function spaces*, Doctoral Dissertation, Nara Women's University, 1993.
- [5] ———, *Pointwise multipliers*, Memoirs of The Akashi College of Technology, 37 (1995), 85–94.
- [6] R. O'Neil, *Convolution operators and  $L(p, q)$  spaces*, Duke Math. J. 30 (1963) 129–142.

## ローレンツ空間における各点的マルチプライヤー

なか い えい いち  
中 井 英 一

大阪教育大学数学教育講座

ルベグ可積分関数空間  $L^p$  ( $0 < p \leq \infty$ ) は完備な擬ノルム空間である。関数  $g$  が  $L^{p_1}$  から  $L^{p_2}$  への各点的マルチプライヤーであるとは、各々の  $f \in L^{p_1}$  に対して関数と関数の各点ごとの積  $fg$  が  $L^{p_2}$  の元になることをいう。 $L^{p_1}$  から  $L^{p_2}$  への各点的マルチプライヤーの全体を  $PWM(L^{p_1}, L^{p_2})$  で表す。このとき、次のことが知られている。もし  $1/p_1 + 1/p_3 = 1/p_2$  ならば、

$$PWM(L^{p_1}, L^{p_2}) = L^{p_3} \quad \text{かつ} \quad \|g\|_{\text{op}} = \|g\|_{L^{p_3}},$$

ここで  $\|g\|_{\text{op}}$  は  $g \in PWM(L^{p_1}, L^{p_2})$  の作用素ノルムを表す。この論文の目的は、この2つの等式を完備な擬ノルム空間であるローレンツ空間に一般化することである。