

# Fractional integrals on martingale Hardy spaces for $0 < p \leq 1$

SADASUE Gaku

Department of Mathematics, Osaka Kyoiku University  
4-698-1 Asahigaoka, Kashiwara, Osaka 582-8582, Japan

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For dyadic martingales,  $H_p$ - $H_q$  boundedness of fractional integrals are pointed out by Chao and Ombe [3]. The purpose of this paper is to extend this to more general martingales in case where  $0 < p \leq 1$ .

**Key Words:** fractional integral, martingale Hardy space, atomic decomposition

## I Introduction

In [3], Chao and Ombe introduced the fractional integrals for dyadic martingale. They pointed out that  $H_p$ - $H_q$  boundedness hold for the fractional integrals.

Recently, the notion of fractional integrals was extended to more general martingales in [6]. They studied the  $L_p$ - $L_q$  boundedness of the fractional integrals, but did not show the  $H_p$ - $H_q$  boundedness for  $0 < p \leq 1$ .

In this note, we will show that  $H_p$ - $H_q$  boundedness of the fractional integrals also holds when  $0 < p \leq 1$ .

The proof is based on the atomic decomposition theorem for martingale Hardy spaces. We also use a method based on a pointwise estimate of fractional integrals, similar to the method in [6].

The organization of this note is as follows. In Section 2, we introduce some notation and give the definition of fractional integrals. In Section 3, we give a proof of  $H_p$ - $H_q$  boundedness of the fractional integrals.

## II Definitions and notations

Let  $(\Omega, \mathcal{F}, P)$  be a probability space, and  $\{\mathcal{F}_n\}_{n \geq 0}$  a nondecreasing sequence of sub- $\sigma$ -algebras of  $\mathcal{F}$  such that  $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$ . For the sake of simplicity, let  $\mathcal{F}_{-1} = \mathcal{F}_0$ . In this note we always suppose that every  $\sigma$ -algebra  $\mathcal{F}_n$  is generated by countable atoms. Denote by  $A(\mathcal{F}_n)$  the set of the atoms of the  $\sigma$ -algebra  $\mathcal{F}_n$ .

We say that the stochastic basis  $\{\mathcal{F}_n\}_{n \geq 0}$  is regular if there exists a constant  $R \geq 2$  such that

$$(2.1) \quad f_n \leq R f_{n-1}$$

holds for all nonnegative martingales  $(f_n)_{n \geq 0}$ .

The expectation operator and the conditional expectation operators relative to  $\mathcal{F}_n$  are denoted by  $E$  and  $E_n$ , respectively. Let  $\mathcal{M}$  be the set of all martingale  $f = (f_n)_{n \geq 0}$  relative to  $\{\mathcal{F}_n\}_{n \geq 0}$  such that  $f_0 = 0$ . For  $f \in \mathcal{M}$ , denote its martingale difference by  $d_n f = f_n - f_{n-1}$  ( $n \geq 0$ , with convention  $d_0 f = 0$ ). Then the maximal function  $f_n^*$  and  $f^*$  are defined by

$$f_n^* = \sup_{0 \leq m \leq n} |f_m|, \quad f^* = \sup_{n \geq 0} |f_n|.$$

For  $\alpha > 0$ , we define the fractional integral  $I_\alpha f = ((I_\alpha f)_n)_{n \geq 0}$  of  $f$  by

$$(I_\alpha f)_n = \sum_{k=1}^n b_{k-1}^\alpha d_k f,$$

where  $b_k$  is an  $\mathcal{F}_k$ -measurable function such that

$$b_k(\omega) = P(B) \quad \text{for } \omega \in B, \quad B \in A(\mathcal{F}_k).$$

This definition of  $I_\alpha$  is introduced in [6], as an extension of one in [8, 3] which is for dyadic martingales.  $I_\alpha$  is a martingale transform introduced by Burkholder [1].

We now introduce martingale Hardy spaces. Denote by  $\Lambda$  the collection of all sequences  $(\lambda_n)_{n \geq 0}$  of nondecreasing, nonnegative and adapted functions, and set  $\lambda_\infty = \lim_{n \rightarrow \infty} \lambda_n$ . For  $f \in \mathcal{M}$ , let

$$\Lambda[\mathcal{P}_p](f) = \{(\lambda_n)_{n \geq 0} \in \Lambda : |f_n| \leq \lambda_{n-1} \ (n \geq 1), \ \lambda_\infty \in L_p\}.$$

Then we define martingale Hardy spaces as follows. For  $p > 0$ , let

$$\begin{aligned} H_p^* &= \{f \in \mathcal{M} : \|f\|_{H_p^*} = \|f^*\|_p < \infty\}, \\ \mathcal{P}_p &= \{f \in \mathcal{M} : \|f\|_{\mathcal{P}_p} = \inf_{(\lambda_n)_{n \geq 0} \in \Lambda[\mathcal{P}_p](f)} \|\lambda_\infty\|_p < \infty\}, \end{aligned}$$

where  $\|\cdot\|_p$  denotes the  $L_p$  norm. We note that if  $\{\mathcal{F}_n\}_{n \geq 0}$  is regular, then  $H_p^* = \mathcal{P}_p$ . See [10], Theorem 2.22.

We introduce the notion of  $(p, \infty)^*$ -atoms. We call an integrable function  $a$  is  $(p, \infty)^*$ -atom if there exists a stopping time  $\nu$  such that

- (a1)  $a_n = E_n[a] = 0$  if  $\nu \geq n$ ,
- (a2)  $\|a\|_\infty \leq P(\nu < \infty)^{-1/p}$ .

### III Hp-Hq boundedness

In this section, we give a proof for the following theorem.

**Theorem 3.1.** *Let  $\{\mathcal{F}_n\}_{n \geq 0}$  be regular, and  $(\Omega, \mathcal{F}, P)$  be non-atomic. Let  $0 < p \leq 1$ ,  $p < q$  and  $\alpha = 1/p - 1/q$ . Then, there exists a constant  $C$  such that*

$$(3.1) \quad \|I_\alpha f\|_{H_q^*} \leq C \|f\|_{H_p^*}$$

for all  $f \in H_p^*$ .

For the proof of Theorem 3.1, we prepare some lemmas.

**Lemma 3.2.** *Let  $\{\mathcal{F}_n\}_{n \geq 0}$  be regular. Then every sequence*

$$B_0 \supset B_1 \supset \cdots \supset B_n \supset \cdots, \quad B_n \in A(\mathcal{F}_n),$$

*has the following property: For each  $n \geq 1$ ,*

$$B_n = B_{n-1} \text{ or } (1 + 1/R)P(B_n) \leq P(B_{n-1}) \leq RP(B_n),$$

*where  $R$  is the constant in (2.1).*

For the proof of Lemma 3.2, see [6], Lemma 3.2.

We next show that  $\{I_\alpha\}_{\alpha > 0}$  is a semigroup.

**Lemma 3.3.** *For  $\alpha, \beta > 0$  and  $f = (f_n)_{n \geq 0} \in \mathcal{M}$ , we have*

$$(3.2) \quad I_{\alpha+\beta} f = I_\alpha(I_\beta f).$$

*Proof.* The proof is a direct computation as follows:

$$\begin{aligned}
I_\alpha(I_\beta f)_n &= \sum_{k=1}^n b_{k-1}^\alpha d_k(I_\beta f) \\
&= \sum_{k=1}^n b_{k-1}^\alpha (b_{k-1}^\beta d_k f) \\
&= \sum_{k=1}^n b_{k-1}^{\alpha+\beta} d_k f = (I_{\alpha+\beta} f)_n.
\end{aligned}$$

The next lemma is a key estimate for the proof of Theorem 3.1.

**Lemma 3.4.** *Let  $(f_n)_{n \geq 0}$  be regular,  $0 < \alpha < 1$ , and  $f = (f_n)_{n \geq 0} \in \mathcal{M}$ . Let  $R$  be the positive constant in (2.1). Suppose that there exists  $B \in \mathcal{F}$  such that  $f^* \leq \chi_B$ . Then,*

$$\begin{aligned}
\text{(i)} \quad & \left| \sum_{k=1}^m b_{k-1}^\alpha d_k f \right| \leq \frac{(R+1)P(B)}{1 - (1+1/R)^{\alpha-1}} b_{m-1}^{\alpha-1} \chi_B \quad \text{for } m \geq 1. \\
\text{(ii)} \quad & \left| \sum_{k=n+1}^m b_{k-1}^\alpha d_k f \right| \leq 2b_n^\alpha \chi_B \quad \text{for } 0 \leq n < m.
\end{aligned}$$

*Proof.* We first show that

$$(3.3) \quad |f_m| \leq \frac{P(B)}{b_m} \chi_B$$

for  $m \geq 0$ . Let  $B' \in A(\mathcal{F}_m)$ . Since  $f_m$  is  $\mathcal{F}_m$  measurable, we have

$$(3.4) \quad f_m = \frac{1}{P(B')} \int_{B'} f_m dP = \frac{1}{b_m} \int_{B'} f_m dP \quad \text{on } B'.$$

Moreover, using the assumption  $f^* \leq \chi_B$ , we have

$$(3.5) \quad \left| \int_{B'} f_m dP \right| \leq \int_{B'} |f_m| dP \leq \int f^* dP \leq P(B).$$

Combining (3.4), (3.5) and  $|f_m| \leq f^* \leq \chi_B$ , we obtain (3.3).

We now show (i). Let  $B' \in A(\mathcal{F}_m)$ . Then, we can take  $B' \in A(\mathcal{F}_m)$  such that  $B' = B_m \subset B_{m-1} \subset \cdots \subset B_0$ . We set  $K = \{1 \leq k \leq m; B_{k-1} \neq B_k\} = \{k_1, k_2, \dots, k_\ell\}$ , where  $k_1 < k_2 < \cdots < k_\ell$ . We use the convention  $k_0 = 0$ . By Lemma 3.2, we have

$$b_{k_j-1} \geq (1+1/R)^{(\ell-j)} b_{m-1} \quad \text{on } B'.$$

Since  $0 < \alpha < 1$ , we have

$$b_{k_j-1}^{\alpha-1} \leq (1+1/R)^{(\alpha-1)(\ell-j)} b_{m-1}^{\alpha-1} \quad \text{on } B'.$$

On the other hand, since  $(f_n)_{n \geq 0}$  is regular, we have

$$|d_k f| = |f_k - f_{k-1}| \leq \left( \frac{1}{b_k} + \frac{1}{b_{k-1}} \right) P(B) \chi_B \leq \frac{(R+1)P(B)}{b_{k-1}} \chi_B$$

by (3.3). Therefore, for  $\omega \in B'$ , we obtain

$$\begin{aligned}
\left| \sum_{k=1}^m b_{k-1}(\omega)^\alpha d_k f(\omega) \right| &= \left| \sum_{j=1}^\ell b_{k_{j-1}}(\omega)^\alpha d_{k_j} f(\omega) \right| \\
&\leq (R+1)P(B) \sum_{j=1}^\ell b_{k_{j-1}}(\omega)^{\alpha-1} \chi_B(\omega) \\
&\leq (R+1)P(B) \left( \sum_{j=1}^\ell (1+1/R)^{(\alpha-1)(\ell-j)} \right) b_{m-1}(\omega)^{\alpha-1} \chi_B(\omega) \\
&\leq \frac{(R+1)P(B)}{1 - (1+1/R)^{\alpha-1}} b_{m-1}(\omega)^{\alpha-1} \chi_B(\omega).
\end{aligned}$$

Thus, we have obtained (i).

We can show (ii) by the following way:

$$\begin{aligned}
\left| \sum_{k=n+1}^m b_{k-1}^\alpha d_k f \right| &= \left| \sum_{k=n+1}^m b_{k-1}^\alpha f_k - \sum_{k=n+1}^m b_{k-1}^\alpha f_{k-1} \right| \\
&= \left| \sum_{k=n+1}^{m-1} (b_{k-1}^\alpha - b_k^\alpha) f_k + b_{m-1}^\alpha f_m - b_n^\alpha f_n \right| \\
&\leq \sum_{k=n+1}^{m-1} (b_{k-1}^\alpha - b_k^\alpha) f^* + b_{m-1}^\alpha f^* + b_n^\alpha f^* \\
&= 2b_n^\alpha f^* \\
&\leq 2b_n^\alpha \chi_B.
\end{aligned}$$

The proof of Lemma 3.4 is completed.

In the next lemma, we regard  $(p, \infty)^*$ -atom  $a$  as a martingale by  $a = (E_n[a])_{n \geq 0}$ , and consider the fractional integral  $I_\alpha a$ .

**Lemma 3.5.** *Let  $\{\mathcal{F}_n\}_{n \geq 0}$  be regular,  $\alpha > 0$ ,  $f = (f_n)_{n \geq 0} \in \mathcal{M}$ . Let  $R$  be the positive constant in (2.1). Suppose that there exists  $B \in \mathcal{F}$  such that  $f^* \leq \chi_B$ . Then, there exists a positive constant  $C_\alpha$  independent of  $f$  and  $B$  such that*

$$(3.6) \quad (I_\alpha f)^* \leq C_\alpha P(B)^\alpha \chi_B.$$

In particular,

$$(3.7) \quad \|I_\alpha a\|_{H_q^*} \leq C_\alpha \quad \text{for all } (p, \infty)^*\text{-atom } a,$$

where  $0 < p < q$ ,  $\alpha = 1/p - 1/q$  and  $C_\alpha$  is the same constant as in (3.6).

*Proof.* We first show (3.6) for  $0 < \alpha < 1$ . We set  $C_\alpha = 2 + (R+1)/[1 - (1+1/R)^{\alpha-1}]$  for  $0 < \alpha < 1$ . Then, we need only to show that

$$(3.8) \quad |(I_\alpha f)_m| \leq C_\alpha P(B)^\alpha \chi_B$$

for every  $m \geq 1$ . To show (3.8), we define a function  $n$  on  $\Omega$  by

$$n(\omega) = \begin{cases} \inf\{k; b_k(\omega) < P(B)\} & (\{k; b_k(\omega) < P(B)\} \neq \emptyset), \\ +\infty & (\{k; b_k(\omega) < P(B)\} = \emptyset). \end{cases}$$

If  $m \leq n(\omega)$ , then by (i) in Lemma 3.4,

$$\begin{aligned}
(3.9) \quad |(I_\alpha f)_m(\omega)| &= \left| \sum_{k=1}^m b_{k-1}(\omega)^\alpha d_k f(\omega) \right| \\
&\leq \frac{(R+1)P(B)}{1 - (1+1/R)^{\alpha-1}} b_{m-1}(\omega)^{\alpha-1} \chi_B(\omega) \\
&\leq \frac{(R+1)P(B)}{1 - (1+1/R)^{\alpha-1}} P(B)^{\alpha-1} \chi_B(\omega) \leq C_\alpha P(B)^\alpha \chi_B(\omega).
\end{aligned}$$

If  $m \leq n(\omega)$ , then by (3.9) and (ii) in Lemma 3.4,

$$\begin{aligned}
 (3.10) \quad |(I_\alpha f)_m(\omega)| &= \left| \sum_{k=1}^{n(\omega)} b_{k-1}(\omega)^\alpha d_k f(\omega) + \sum_{k=n(\omega)+1}^m b_{k-1}(\omega)^\alpha d_k f(\omega) \right| \\
 &\leq \frac{R+1}{1 - (1+1/R)^{\alpha-1}} P(B)^\alpha \chi_B(\omega) + 2b_{n(\omega)}(\omega)^\alpha \chi_B(\omega) \\
 &\leq \frac{R+1}{1 - (1+1/R)^{\alpha-1}} P(B)^\alpha \chi_B(\omega) + 2P(B)^\alpha \chi_B(\omega) \\
 &= C_\alpha P(B)^\alpha \chi_B(\omega).
 \end{aligned}$$

Thus, we have obtained (3.6) for  $0 < \alpha < 1$ .

For  $\alpha \geq 1$ , we take an integer  $\ell$  such that  $\alpha = \ell/2 + \beta$ , where  $0 \leq \beta < 1/2$ . Since we have obtained (3.6) for  $0 < \alpha < 1$ ,  $I_\beta f$  and  $I_{1/2} f$  satisfy  $(I_\beta f)^* \leq C_\beta P(B)^\beta \chi_B$  and  $(I_{1/2} f)^* \leq C_{1/2} P(B)^{1/2} \chi_B$  respectively. Combining these inequalities and (3.2), we have

$$(I_\alpha f)^* = \{I_\beta \circ (I_{1/2})^\ell f\}^* \leq C_{1/2}^\ell C_\beta P(B)^{\ell/2+\beta} \chi_B.$$

We set  $C_\alpha = C_{1/2}^\ell C_\beta$ . Then, we have obtained (3.6) for  $\alpha \geq 1$ .

We now show (3.7). Let  $a$  be a  $(p, \infty)^*$ -atom. For the stopping time  $\nu$  in the condition (a2), we set  $B_\nu = \{\nu < \infty\}$ . Then, we have  $a^* \leq P(B_\nu)^{-1/p} \chi_{B_\nu}$ . Therefore, by (3.6) we have

$$(I_\alpha a)^* \leq C_\alpha P(B_\nu)^\alpha P(B_\nu)^{-1/p} \chi_{B_\nu} = C_\alpha P(B_\nu)^{-1/q} \chi_{B_\nu}.$$

Hence, we have

$$\begin{aligned}
 \|I_\alpha a\|_{H_q^*}^q &= E[\{(I_\alpha a)^*\}^q] \\
 &\leq C_\alpha^q P(B_\nu)^{-1} E[\chi_{B_\nu}] \\
 &= C_\alpha^q.
 \end{aligned}$$

Thus, we have obtained (3.7).

To prove Theorem 3.1, we recall the atomic decomposition theorem for  $\mathcal{P}_p$ .

**Theorem 3.6.** *For  $f \in \mathcal{P}_p$ , there exist a sequence of  $(p, \infty)^*$ -atoms  $(a^k)_{k \in \mathbb{Z}}$  and  $\mu = (\mu_k)_{k \in \mathbb{Z}} \in \ell^p$  such that*

$$(3.11) \quad f_n = \sum_k \mu_k a_n^k$$

for all  $n \in \mathbb{N}$ . Moreover, if  $0 < p \leq 1$ , there exists a positive constant  $K$  independent of  $f$  and  $a^k$ ,  $\mu_k$  such that

$$K^{-1} \|f\|_{\mathcal{P}_p} \leq \inf \|\mu\|_{\ell_p} \leq K \|f\|_{\mathcal{P}_p},$$

where the infimum is taken over all decompositions of  $f$  of the form (3.11).

For the proof of Theorem 3.6, see [10], Theorem 2.3.

We now give the proof of Theorem 3.1.

*Proof of Theorem 3.1.*

Let  $f \in H_p^*$ . Since  $\{\mathcal{F}_n\}_{n \geq 0}$  is regular, we have  $H_p^* = \mathcal{P}_p$ . Hence, by Theorem 3.6, there exist a sequence of  $(p, \infty)^*$ -atoms  $(a^k)_{k \in \mathbb{Z}}$ ,  $\mu = (\mu_k)_{k \in \mathbb{Z}} \in \ell^p$  and a positive constant  $K$  such that

$$(3.12) \quad f_n = \sum_k \mu_k a_n^k \quad (n \in \mathbb{N}) \quad \text{and} \quad K^{-1} \|f\|_{H_p^*} \leq \|\mu\|_{\ell_p} \leq K \|f\|_{H_p^*}.$$

Combining (3.12) and (3.7), we can prove Theorem 3.1. Indeed, if  $q \geq 1$ , then we have

$$\begin{aligned}
\|I_\alpha f\|_{H_q^*} &= \|(I_\alpha f)^*\|_q \\
&\leq \left\| \sum_k |\mu_k| (I_\alpha a^k)^* \right\|_q \\
&\leq \sum_k |\mu_k| \|(I_\alpha a^k)^*\|_q \\
&\leq C_\alpha \sum_k |\mu_k| \\
&\leq C_\alpha \|\mu\|_{\ell_p} \\
&\leq KC_\alpha \|f\|_{H_p^*}.
\end{aligned}$$

If  $q < 1$ , then we have

$$\begin{aligned}
\|I_\alpha f\|_{H_q^*}^q &= \|(I_\alpha f)^*\|_q^q \\
&\leq \left\| \sum_k |\mu_k| (I_\alpha a^k)^* \right\|_q^q \\
&\leq \sum_k |\mu_k|^q \|(I_\alpha a^k)^*\|_q^q \\
&\leq C_\alpha^q \sum_k |\mu_k|^q \\
&= C_\alpha^q \|\mu\|_{\ell_q}^q \\
&\leq C_\alpha^q \|\mu\|_{\ell_p}^q \\
&\leq K^q C_\alpha^q \|f\|_{H_p^*}^q.
\end{aligned}$$

The proof of Theorem 3.1 is completed.

## References

- [1] D. L. Burkholder, Martingale transforms. *Ann. Math. Stat.*, 37 (1966), 1494–1504.
- [2] D. L. Burkholder and R. F. Gundy, Extrapolation and interpolation of quasi-linear operators on martingales, *Acta Math.* 124 (1970), 249–304.
- [3] J.-A. Chao and H. Ombe, Commutators on Dyadic Martingales, *Proc. Japan Acad.*, 61, Ser. A (1985), 35–38.
- [4] L. I. Hedberg, On certain convolution inequalities, *Proc. Amer. Math. Soc.* 36 (1972), 505–510.
- [5] S. G. Krantz, Fractional integration on Hardy spaces, *Studia Math.* 73 (1982), no. 2, 87–94.
- [6] E. Nakai and G. Sadasue, Martingale Morrey-Campanato spaces and fractional integrals, Preprint.
- [7] J. Neveu, *Discrete-parameter martingales*, North-Holland, Amsterdam, 1975.
- [8] C. Watari, Multipliers for Walsh Fourier series. *Tohoku Math. J.*, 16 (1964), 239–251.
- [9] F. Weisz, Martingale Hardy spaces for  $0 < p \leq 1$ . *Probab. Theory Related Fields* 84 (1990), no. 3, 361–376.
- [10] F. Weisz, *Martingale Hardy spaces and their applications in Fourier analysis*, Lecture Notes in Mathematics, 1568, Springer-Verlag, Berlin, 1994.

## 指数1 以下のマルティンゲールハーディー空間における分数べき積分

さだ すえ がく  
貞 末 岳

数学教育講座

2 進マルティンゲールハーディー空間においては、分数べき積分の  $H_p - H_q$  有界性が知られている。本論文では、指数  $0 < p \leq 1$  のとき、そのことがより一般のマルティンゲールにおいて成り立つことを示す。

キーワード：分数べき積分，マルティンゲールハーディー空間，アトムック分解